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Online Exact Differentiation and Notion of Asymptotic Algebraic Observers

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***Abstract*—In recent years, the availability of computer-based methods has created a revival of interests in exploring algebraic methods in nonlinear context. This note proposes a new approach to algebraic nonlinear observer design. After giving the notion of algebraic observability, and based on a novel algorithm of exact differentiation, the formulation of the nonlinear observer is realized via the construction of a set of linear time-varying differentiators. An example of a chemical reaction is given to show the effectiveness of our approach.**

***Index Terms*—Exact differentiation, nonlinear observers, system theory, time-varying systems.**

I. Introduction

Nonlinear observer design has received considerable attention since the appearance of the pioneer works of Kalman [1] and Luenberger [2]. The available techniques for the design of nonlinear observer can be classified in three groups. First, high-gain observers based on pole-placement algorithms as in [3], algebraic Ricatti equation (ARE) as in [4]–[8], Lyapunov equation as in [9]–[12], and backstepping method as in [13]. Second, Kalman filter based observers, whose design is based on local linearization of the system around a reference trajectory, restricting the validity of the approach within a small region of the state space [14]. Third, observers with input and output injection terms as in [15]–[19]. Some of these observers necessitate estimation of the output derivatives and no complete analysis of the observer design has been exposed. Furthermore, the linearization approaches based upon coordinate transformation, is generally based upon a set of extremely restrictive conditions, that may hardly be met by any physical system.

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Motivated by some results on algebraic observability, Diop *et al.* [20] proposed a general observer design methodology based upon numerical differentiation and the interpretation of observability of a system as the solvability of the system’s dynamical equations for the state vector in terms of a finite number of derivatives of the output and input. This idea has been developed and performed by Ibrir and Diop in [21]. In [4], [22], and [23], Ibrir has proposed a set of linear continuous-time differentiators to build a new kind of state estimators, called algebraic observers. Due to the residual error provided by the differentiator system, the estimated states could never converge to the exact ones, but they approach the true ones by regulating an adjustable parameter.

In this note, we propose a new approach to observation of nonlinear systems that met the algebraic observability conditions. The proposed observer enjoys the advantages: 1) of being free of some restrictive conditions as Lipschitz condition or Hölder continuity condition; 2) always exists whenever the algebraic observability condition is verified; 3) insensible to error modeling while systems being observed are given in classical Brunovski forms. The formulation of the nonlinear observer is quite new and is based upon the construction of a set of linear time-varying (LTV) differentiation systems. The novelty of the proposed observer is that the observer states are given in term of a static diffeomorphism that involves the states of the LTV systems. Hence, the observation problem becomes less dependent to the form of nonlinearities and more attached to the calculation of the time-derivatives of the inputs and the outputs. Therefore, we can say that the key element, in this kind of observer design, is the accuracy of the selected differentiation method and its robustness. It is well known that the differentiation problem of signals is an old and challenging problem. Numerous techniques are known to be efficient for the estimation of the few first derivatives from data with low frequency content, such as polynomial- and spline-based least squares, and averaged central differences [22]. The main advantages of such observers are intuitiveness, flexibility and speed. However, as is the case of many inverse problems, differentiation is an ill-posed operator. In this case, the use of regularization to partially overcome the noise sensitivity is recommended [22]. Other concepts of signal differentiation have been formulated by the use of high-gain observers, see [4], [23]–[25]. Finite-time differentiators have been proposed in [26]. Unfortunately, the majority of these techniques are not able to furnish exact output derivatives, and others are in need of some information such as the upper bounds of the higher-order derivatives [26].

In order to master the crucial point of the differentiation problem, comply with the existing practical situations, and ensure certain reliability while estimating the slopes of outputs, issued from different control inputs, we propose, for the first time, a novel exact differentiator whose states converge asymptotically to the successive higher derivatives of a given input signal. This differentiator does not need any information about the signal to be differentiated, like the nature of the signal or *a priori* knowledge of the upper bounds of its higher derivatives. Since all the derivatives converge asymptotically to the true ones, then any state given in term of a static diffeomorphism, involving these derivatives, will be reproduced exactly with the same convergence rate of the derivative estimates.

In Section II, the theory of the LTV differentiator is exposed. Section III is devoted to the design methodology of the asymptotic algebraic observer. The combination of the asymptotic algebraic observer with the classical Luenberger observer will be the subject of Section IV. Finally, the note ends with some concluding remarks. Throughout this note, we note $\|\cdot\|$ the classical Euclidean norm, $\circ$: is the usual composition operator of functions, $\mathbb{R}_{\geq 0}$ stands for the set of positive real numbers, $y^{(i)}$ is the ith time-derivative of y with respect to t such that $y^{(0)} = y \cdot \lambda_{\min}(A)$ is the smallest eigenvalue of the matrix A and $\lambda_{\max}(A)$ denotes the largest eigenvalue of A. A' is the transpose of $A \cdot \stackrel{\text{def}}{=}$ stands for equality, by definition. C_n^k stands for the binomial coefficient. We call that a function $r: \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is a K-function if it is continuous, strictly increasing and $r(0) = 0$. A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ is a KL-function if for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is a K-function, and for each fixed $s \geq 0$ it is decreasing to zero as $t \to \infty$.

II. LTV Differentiator

Before giving the main result of this note, we need to give the following lemma.

Lemma 1: Let $f(t): \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ be a uniformly bounded and a continuously differentiable function for all $t \geq 0$, then

$$\text{i)} \lim_{t\to\infty} e^{-t^2} \int_0^t e^{\tau^2} f(\tau) d\tau = 0 \tag{1}$$

and

$$\text{ii)} \lim_{t\to\infty} e^{-t^2} \int_0^t f(\tau) \left(\int_0^\tau e^{\zeta^2} d\zeta \right) d\tau = 0. \tag{2}$$

Proof: i) Since $f(t)$ is uniformly bounded, we have

$$C_1 e^{-t^2} \int_0^t e^{\tau^2} d\tau \leq e^{-t^2} \int_0^t e^{\tau^2} f(\tau) d\tau \leq C_2 e^{-t^2} \int_0^t e^{\tau^2} d\tau \tag{3}$$

where $C_1 = \min_{t\geq 0}(f(t))$, and $C_2 = \max_{t\geq 0}(f(t))$. The integral $e^{-t^2} \int_0^t e^{\tau^2} d\tau$ is called the Dawson's integral which vanishes to zero when $t \to \infty$. To prove (1), we develop the expansions of the Dawson's integral for large values of t. Remark that for a fixed value b

$$e^{-t^2} \int_0^b e^{\tau^2} d\tau < e^{-t^2} \cdot e^b \cdot b \simeq O\left(e^{-t^2}\right); t \to \infty.$$

Thus, it yields a small contribution for large values of t. The first two terms in the asymptotic expansion of $e^{-t^2} \int_b^t e^{\tau^2} d\tau$ are $1/2t + 1/4t^3 + O\left(1/t^5\right); t \to \infty$. If we continue the integration process, one can obtain the following:

$$\lim_{t\to\infty} e^{-t^2} \int_0^t e^{\tau^2} d\tau = \lim_{t\to\infty} \left(\frac{1}{2t} + \sum_{i=1}^{\infty} \frac{a_i}{2^{i+1} t^{2i+1}} \right) = 0 \tag{4}$$

where $a_0 = 1$ and $a_i = (2i-1)a_{i-1}, \ \forall\, i$. *ii)* We have

$$C_1 e^{-t^2} \int_0^t \left(\int_0^\tau e^{\zeta^2} d\zeta \right) d\tau \leq e^{-t^2} \int_0^t f(\tau) \left(\int_0^\tau e^{\zeta^2} d\zeta \right) d\tau \leq C_2 e^{-t^2} \int_0^t \left(\int_0^\tau e^{\zeta^2} d\zeta \right) d\tau. \tag{5}$$

Since $e^{-\tau^2} \int_0^\tau e^{\zeta^2} d\zeta \leq k$, where $0 < k < 1$; see [27, pp. 297–319], then

$$\lim_{t\to\infty} e^{-t^2} \int_0^t \left(\int_0^\tau e^{\zeta^2} d\zeta \right) d\tau = \lim_{t\to\infty} k e^{-t^2} \int_0^t e^{\tau^2} d\tau = 0. \tag{6}$$

Consequently, (2) is verified. Now, we are ready to introduce the basic result of this note.

A. Differentiator Design

Theorem 1: Let $\xi(t): \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ be a scalar continuous function of class C^∞ and let $(\rho_i, \ i = 0, 1, 2, \ldots)$ be a sequence of positive real numbers. If the higher derivatives of $\xi(t)$ satisfy

$$\sup_{t\geq 0} \left| \xi^{(i)}(t) \right| \leq \rho_i, \qquad i = 0, 1, 2, \ldots \tag{7}$$

then the state vector of the following time-varying system:

$$\dot{x}(t) = A(t)x(t) + B(t)\xi(t) \tag{8}$$

converges to the vector $[\, \xi(t) \quad \dot{\xi}(t) \,]'$ when time elapses. We note

$$A(t) = \begin{bmatrix} 0 & 1 \\ -\alpha^2 t^2 & -2\alpha t \end{bmatrix} \quad B(t) = \begin{bmatrix} 0 \\ \alpha^2 t^2 \end{bmatrix}, \ \ \alpha \in \mathbb{R}_{>0}. \tag{9}$$

Proof: To prove this result, it is sufficient to show that $\lim_{t\to\infty} x_1(t) = \xi(t)$. The state variable $x_1(t)$ verifies the differential equation

$$\ddot{x}_1(t) + 2\alpha t \dot{x}_1(t) + \alpha^2 t^2 \left(x_1(t) - \xi(t)\right) = 0. \tag{10}$$

Equation (10) is transformed to an ordinary differential equation with constant coefficients by taking the Liouville–Green transformation $y(t) \stackrel{\text{def}}{=} x_1(t) e^{(\alpha/2t^2)}$ which gives

$$\ddot{y}(t) - \alpha y(t) = \alpha^2 t^2 e^{1/2\alpha t^2} \xi(t). \tag{11}$$

Using the method of variation of parameters, we obtain

$$\begin{aligned} x_1(t) =& c_1 e^{-(1/2)t\left(\alpha t + 2\sqrt{\alpha}\right)} + c_2 e^{(1/2)t\left(-\alpha t + 2\sqrt{\alpha}\right)} \\ &- \frac{1}{2}\alpha^{3/2} e^{-(1/2)t\left(\alpha t + 2\sqrt{\alpha}\right)} \\ &\times \underbrace{\int t^2 e^{(1/2)t\left(\alpha t + 2\sqrt{\alpha}\right)} \xi(t) dt}_{I_1} \\ &+ \frac{1}{2}\alpha^{3/2} e^{(1/2)t\left(-\alpha t + 2\sqrt{\alpha}\right)} \\ &\times \underbrace{\int t^2 e^{-(1/2)t\left(-\alpha t + 2\sqrt{\alpha}\right)} \xi(t) dt}_{I_2} \end{aligned} \tag{12}$$

where c_1 and c_2 are real constants. Define $v(t) \stackrel{\text{def}}{=} \int t^2 e^{(1/2)t\left(\alpha t + 2\sqrt{\alpha}\right)} dt$, $v^\star(t) \stackrel{\text{def}}{=} \int t^2 e^{-(1/2)t\left(-\alpha t + 2\sqrt{\alpha}\right)} dt$, $\phi_1(t) \stackrel{\text{def}}{=} - 1/2\alpha^{3/2} e^{-(1/2)t\left(\alpha t + 2\sqrt{\alpha}\right)}$, $\phi_2(t) \stackrel{\text{def}}{=} 1/2\alpha^{3/2} e^{(1/2)t\left(-\alpha t + 2\sqrt{\alpha}\right)}$, $\eta_1 \stackrel{\text{def}}{=} 1/\sqrt{2}\,(\sqrt{\alpha}t + 1)$, $\eta_2 \stackrel{\text{def}}{=} 1/\sqrt{2}\,(\sqrt{\alpha}t - 1)$. By integrating I_1 and I_2 by part, then $I_1 = v(t)\xi(t) - \int v(t)\dot{\xi}(t)dt$, and $I_2 = v^\star(t)\xi(t) - \int v^\star(t)\dot{\xi}(t)dt$. We have

$$v(t) = \frac{\left(e^{(1/2)t\left(\alpha t + 2\sqrt{\alpha}\right)} \left(\alpha t - \sqrt{\alpha}\right)\right)}{\alpha^2} \tag{13}$$

and

$$v^\star(t) = \frac{\left(e^{(1/2)t\left(\alpha t - 2\sqrt{\alpha}\right)} \left(\alpha t + \sqrt{\alpha}\right)\right)}{\alpha^2}. \tag{14}$$

Then, using (13) and (14), we have

$$\phi_1 v(t)\xi(t) + \phi_2 v^\star(t)\xi(t) = \xi(t). \tag{15}$$

Consequently

$$\begin{aligned} x_1(t) = \xi(t) &+ c_1 e^{-(1/2)t\left(\alpha t + 2\sqrt{\alpha}\right)} + c_2 e^{(1/2)t\left(-\alpha t + 2\sqrt{\alpha}\right)} \\ &- \phi_1(t) \underbrace{\int v(t)\dot{\xi}(t)dt}_{I_3} - \phi_2(t) \underbrace{\int v^\star(t)\dot{\xi}(t)dt}_{I_4}. \end{aligned} \tag{16}$$

Integrating I_3 and I_4 by part, we obtain $I_3 = \dot{\xi}(t)\int v(t)dt - \int \ddot{\xi}(t)\left(\int v(t)dt\right)dt$, and $I_4 = \dot{\xi}(t)\int v^\star(t)dt - \int \ddot{\xi}(t)\left(\int v^\star(t)dt\right)dt$. We have $\int v(t)dt = 1/\alpha^2 e^{(1/2)t\left(\alpha t + 2\sqrt{\alpha}\right)} - 1/\alpha^{3/2}\int e^{(1/2)t\left(\alpha t+2\sqrt{\alpha}\right)}dt$, and $\int v^\star(t)dt = 1/\alpha^2 e^{(1/2)t\left(\alpha t-2\sqrt{\alpha}\right)} + 1/\alpha^{3/2}\int e^{(1/2)t\left(\alpha t-2\sqrt{\alpha}\right)}dt$. Remark that the quantity

$$-\frac{1}{\alpha^2}\phi_1(t)e^{(1/2)t\left(\alpha t+2\sqrt{\alpha}\right)}\dot{\xi}(t) - \frac{1}{\alpha^2}\phi_2(t)e^{(1/2)t\left(\alpha t-2\sqrt{\alpha}\right)}\dot{\xi}(t)$$

is equal to zero. Moreover

$$\begin{aligned}\lim_{t\to\infty}&\frac{\dot{\xi}(t)}{\alpha^{3/2}}\phi_1(t)\int e^{(1/2)t\left(\alpha t+2\sqrt{\alpha}\right)}dt\\ &= -\lim_{\eta_1\to\infty}\frac{\dot{\xi}(t)}{\sqrt{2\alpha}}e^{-\eta_1^2}\int e^{\eta_1^2}d\eta_1 = 0 \qquad (17)\\ \lim_{t\to\infty}&\frac{\dot{\xi}(t)}{\alpha^{3/2}}\phi_2(t)\int e^{(1/2)t\left(\alpha t-2\sqrt{\alpha}\right)}dt\\ &= \lim_{\eta_2\to\infty}\frac{\dot{\xi}(t)}{\sqrt{2\alpha}}e^{-\eta_2^2}\int e^{\eta_2^2}d\eta_2 = 0. \qquad (18)\end{aligned}$$

To end the proof, it remains to study the limit of the function

$$\phi_1(t)\int\ddot{\xi}(t)\left(\int v(t)dt\right)dt + \phi_2(t)\int\ddot{\xi}(t)\left(\int v^\star(t)dt\right)dt \qquad (19)$$

when $t\to\infty$. Term (19) can be written as

$$\begin{aligned}&-\frac{1}{\sqrt{2}\alpha}e^{-\eta_1^2}\int e^{\eta_1^2}\ddot{\xi}(t)d\eta_1\\ &+\frac{1}{\sqrt{2}\alpha}e^{-\eta_2^2}\int e^{\eta_2^2}\ddot{\xi}(t)d\eta_2\\ &+\frac{1}{\alpha}e^{-\eta_1^2}\int\ddot{\xi}(t)\left(\int e^{\eta_1^2}d\eta_1\right)d\eta_1\\ &+\frac{1}{\alpha}e^{-\eta_2^2}\int\ddot{\xi}(t)\left(\int e^{\eta_2^2}d\eta_2\right)d\eta_2. \qquad (20)\end{aligned}$$

Since the higher derivatives of $\xi(t)$ are bounded and using the results of lemma 1, we conclude that the term of (20) vanishes to zero when $t\to\infty$ and, therefore, $\lim_{t\to\infty} x_1(t) = \xi(t)$.

B. Sensitivity

For practical implementation of differentiator (8) it is necessary to saturate the time t, that appears in the expressions of $A(t)$ and $B(t)$. The saturation of the term αt in (8) shall be done when the differentiation error becomes negligible. Let ϵ be an arbitrary small positive number, then we propose to rewrite the dynamics of the differentiator (8) as

$$\dot{x}(t) = \begin{bmatrix}0 & 1\\ -\varphi^2(t) & -2\varphi(t)\end{bmatrix}x(t) + \begin{bmatrix}0\\ \varphi^2(t)\end{bmatrix}\xi(t) \qquad (21)$$

where $\varphi(t)$ is defined by

$$\dot{\varphi}(t)\stackrel{\text{def}}{=}\begin{cases}\alpha & \text{for } |x_1(t)-\xi(t)| > \epsilon\\ 0 & \text{for } |x_1(t)-\xi(t)| \le \epsilon\end{cases} \qquad (22)$$

such that $\varphi(0) = 0$. When $|x_1(t)-\xi(t)| > \epsilon$, the dynamics of differentiator (21) is the same dynamics of (8). For $|x_1(t)-\xi(t)| \le \epsilon$, the function φ becomes time-invariant, i.e., $\varphi = \bar{\varphi}$, where $\bar{\varphi}$ is a positive constant. Consequently, the dynamics of the differentiator (21) [or the dynamics of $x_2(t)$ in (21)] is reduced to an output of a stable time-invariant linear differentiator whose transfer function is $\bar{\varphi}^2 s/(s+\bar{\varphi})^2$ (here, s denotes the Laplace variable). Since the state $x_2(t)$ always represents the first derivative of $x_1(t)$, then computing the difference $|x_1(t)-\xi(t)|$ is a necessary and a sufficient tool to decide about the precision of the differentiation error. Moreover, checking the value of $|x_1(t)-\xi(t)|$ will serve as a practical guide to compute the time-derivative of $\xi(t)$ without any knowledge of the upper bounds of the ξ-derivatives.

III. Definition of the Asymptotic Algebraic Observer

In this note, we will not give explicitly the detailed algorithms for system estimation, but we refer the reader to [28]–[32] to see what have been done in this area. We define the algebraic observability condition as follows.

Definition 1: Consider the nonlinear system described by the following dynamic equations:

$$\begin{cases}\dot{x}(t) = f(x(t), u(t))\\ y(t) = h(x(t))\end{cases} \qquad (23)$$

where $f: \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is continuously differentiable and satisfies $f(0,0) = 0$. $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the input vector, and $y(t)\in\mathbb{R}$ is a smooth nonsingular output. We assume that $y(t)$ and $u(t)$ are continuously differentiable for all $t\ge 0$. System (23) is said to be algebraically observable if there exist two positive integers μ and ν such that

$$x(t) = \phi\left(y, \dot{y}, \ddot{y}, \ldots, y^{(\mu)}, u, \dot{u}, \ddot{u}, \ldots, u^{(\nu)}\right)(t) \qquad (24)$$

where $\phi(\cdot): \mathbb{R}^{\mu+1}\times\mathbb{R}^{(v+1)m}\mapsto\mathbb{R}^n$ is a differentiable vector valued nonlinearity of the inputs, the outputs, and their derivatives.

Notice that the last definition has been introduced in reference [33] to characterize the *uniform complete observability*. Recall that for nonlinear systems, there exists a set of control inputs which renders system (23) unobservable. We refer the reader to [34] for introductory discussions of this problem. For our case, we define this class of bad inputs as follows.

Definition 2: System (23) is algebraically observable for any input, if the vector valued

$$x(t) = \phi\left(y, \dot{y}, \ddot{y}, \ldots, y^{(\mu)}, u, \dot{u}, \ddot{u}, \ldots, u^{(\nu)}\right)(t)$$

is defined on $\mathbb{R}^{\mu+1}\times\mathbb{R}^{(v+1)m}\mapsto\mathbb{R}^n$ for all $u\in U$. We call U the set of continuously differentiable control inputs for which the state vector (24) is defined everywhere, and we note $U^\star$, the set of bad inputs that makes (24) singular.

In order to use the differentiator (8), we are obliged to guarantee the boundedness of the output and its derivatives. For this reason, we introduce the new output

$$\widetilde{y}(t) = \arctan(t)\circ y(t). \qquad (25)$$

The output $y(t)$ may be either bounded or unbounded function of time. We will prove that whatever the nature of y (i.e., bounded or unbounded), the new output $\widetilde{y}(t)$ enjoys the property of being uniformly bounded along with its higher derivatives. For this reason, we distinguish two cases.

Case 1 $y(t)$ Uniformly Bounded: When $y(t)$ is uniformly bounded, then $y(t)$ is a globally Lipschitz, see Appendix for the proof. Using the result of Khalil (see [35, Lemma 2.3, pp. 77–78]) which states that the first time-derivative of a globally Lipschitz function is uniformly bounded function, then with the same analysis we conclude that the second time-derivative of $y(t)$ is a globally Lipschitz if its first derivative does, and so on. Repeating the last argument for the higher time-derivatives of $y(t)$, we deduce that the higher time-derivatives of any uniformly bounded output $y(t)$ are uniformly bounded. Recall that our interest is to prove the uniform boundedness of $\widetilde{y}(t)$. Since any derivative $d^i\widetilde{y}(t)/dt^i = d^{i-1}/dt^{i-1}\left(\dot{y}(t)/1+y^2(t)\right), \forall\, i$ is defined everywhere and is expressed in terms of the derivatives of $y(t)$ which are uniformly bounded, this implies immediately that $\widetilde{y}^{(i)}(t)$, $i = 1, 2, \ldots$ are also uniformly bounded.

Case 2 $y(t)$ Unbounded: Since $y(t)$ is not singular and continuously differentiable (by Definition 1), then whatever the nature of the divergence of $y(t)$ (i.e., a finite-time escape to infinity or slowly monotone divergence), we could write that $\lim_{t\to\infty} y(t) = \pm\infty$. To demonstrate the boundedness of $\widetilde{y}$ in this case, we introduce the following lemma.

Lemma 2: Let $y(t)\colon \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ be a continuous function of class C^{∞} such that $\lim_{t\to\infty} y(t) = \pm\infty$. Then the function $\widetilde{y}(t) = \arctan(t) \circ y(t)$ is uniformly bounded with its derivatives $\widetilde{y}^{(i)}(t), i = 1, 2, \ldots$. Furthermore $\lim_{t\to\infty} \widetilde{y}^{(i)}(t) = 0$ for all $i \geq 1$.

For the proof of this lemma, see Appendix.

A. Change of Coordinate

Because of the uniform-boundedness property of $\widetilde{y}(t)$, it is advantageous to rewrite the state vector x in terms of the $\widetilde{y}$-derivatives. Using the fact that $\dot{y}(t) = \gamma_1(y(t), \dot{\widetilde{y}}(t)) = \left(1 + y^2(t)\right)\dot{\widetilde{y}}(t)$, and $\ddot{y}(t) = \gamma_2(y(t), \dot{\widetilde{y}}(t), \ddot{\widetilde{y}}(t)) = \left(1 + y^2(t)\right)(2y(t)\dot{\widetilde{y}}(t)^2 + \ddot{\widetilde{y}}(t))$, then the higher derivatives of y can be easily computed in terms of the output y and the derivatives of $\widetilde{y}$. In other words, there exists a diffeomorphism $\Gamma = [\,\gamma_1(\cdot) \quad \cdots \quad \gamma_\mu(\cdot)\,]'\,(t)$ such that

$$y^{(i)}(t) = \gamma_i\left(y, \dot{\widetilde{y}}, \ldots, \widetilde{y}^{(i)}\right)(t), \qquad 1 \leq i \leq \mu. \tag{26}$$

Consequently, the state vector x is written in the new coordinates as

$$\begin{aligned} x(t) =& \phi(\cdot,\cdot) \circ \Gamma(\cdot) \\ =& \psi\left(y, \dot{\widetilde{y}}, \ddot{\widetilde{y}}, \ldots, \widetilde{y}^{(\mu)}, u, \dot{u}, \ddot{u}, \ldots, u^{(\nu)}\right)(t). \end{aligned} \tag{27}$$

Remark 1: In order to smooth the higher derivatives of $\widetilde{y}(t)$, it is recommended to take $\widetilde{y}(t) = \arctan(\beta y(t))$, where β is small positive parameter.

In the sequel, all the state variables are time-dependent, and for notation simplicity, the time variable t is omitted. The whole design of the asymptotic algebraic observer for multiple-input–single-output (MISO) nonlinear systems is given in the following theorem.

Theorem 2: Consider (23). If (23) is algebraically observable, then for any $u \in U$ such that y is continuously differentiable, the dynamic system

$$\begin{aligned} \hat{x} &= \psi\left(y, \xi_2, \xi_4, \ldots, \xi_{2\mu}, u, \dot{u}, \ddot{u}, \ldots, u^{(\nu)}\right) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -\alpha^2 t^2 \left(\xi_1 - \arctan(y)\right) - 2\alpha t \xi_2 \\ \dot{\xi}_i &= \xi_{i+1} \\ \dot{\xi}_{i+1} &= -\alpha^2 t^2 \left(\xi_i - \xi_{i-1}\right) - 2\alpha t \xi_{i+1}; \\ i &= 3, 5, 7, \ldots, 2\mu - 1 \end{aligned} \tag{28}$$

is an asymptotic algebraic observer for system (23) where the parameter $\alpha \in \mathbb{R}_{>0}$ is introduced to master the rate of convergence of the derivative estimates.

Proof: We see that (28) is a concatenation of the differentiator given in Theorem 1. System (8) is augmented in order to have the μth derivative of y. Using the results of Theorem 1, we obtain for $(1 \leq i \leq \mu)$ $\lim_{t\to\infty} \widetilde{y}^{(i)} = \xi_{2i}$. Consequently

$$\begin{aligned} \lim_{t\to\infty}\, &\psi\left(y, \xi_2, \xi_4, \ldots, \xi_{2\mu}, u, \dot{u}, \ddot{u}, \ldots, u^{(\nu)}\right) \\ &- \psi\left(y, \dot{\widetilde{y}}, \ddot{\widetilde{y}}, \ldots, \widetilde{y}^{(\mu)}, u, \dot{u}, \ddot{u}, \ldots, u^{(\nu)}\right) = 0. \end{aligned} \tag{29}$$

B. Example Catalyst Batch Reactor

Consider the second order chemical kinetics, coupled with a second order decay rate of the catalyst activity [36]

$$\begin{cases} \dot{x}_1 = -k x_2 x_1^2 \\ \dot{x}_2 = -k_d x_2^2 x_1 \\ y = x_1 \end{cases} \tag{30}$$

where x_1 is the concentration of the reactant, k is the reaction rate constant, x_2 is the catalyst activity, and k_d is the specific decay constant. From (30), we have $x_1 = \phi_1(y) = y$, and $x_2 = \phi_2(y, \dot{y}) = -\dot{y}/(ky^2)$ which implies that x_1 and x_2 are algebraically observable. As we have introduced previously, $\widetilde{y} = \arctan(y)$. Then $\dot{y} = \gamma_1(y, \dot{\widetilde{y}}) = \dot{\widetilde{y}}(1 + y^2)$, which gives

$$\begin{aligned} x_1 &= \psi_1(y) = y \\ x_2 &= \phi_2(\cdot) \circ \gamma_1(\cdot) = \psi_2\left(y, \dot{\widetilde{y}}\right) = -\frac{(1+y^2)\dot{\widetilde{y}}}{ky^2}. \end{aligned} \tag{31}$$

According to (28), the observer is readily constructed as

$$\begin{cases} \hat{x}_2 = -\frac{(1+y^2)\xi_2}{(ky^2)} \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = -\alpha^2 t^2 \left(\xi_1 - \arctan(y)\right) - 2\alpha t \xi_2 \end{cases} \tag{32}$$

where ξ_1 and ξ_2 converge asymptotically to $\widetilde{y}$ and $\dot{\widetilde{y}}$, respectively. As we have mentioned in Section II-B, practical realization of observer (32) needs the saturation of terms αt that appear in the right-hand side of (32). For this purpose, observer (32) is replaced by

$$\begin{cases} \hat{x}_2 = -\frac{(1+y^2)\xi_2}{(ky^2)} \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = -\varphi^2 \left(\xi_1 - \arctan(y)\right) - 2\varphi \xi_2 \\ \dot{\varphi} = \begin{cases} \alpha & \text{if } |\xi_1 - \arctan(y)| > \epsilon \\ 0 & \text{if } |\xi_1 - \arctan(y)| \leq \epsilon \end{cases} \end{cases} \tag{33}$$

where ϵ is any desired precision that seems to be satisfactory in practice. In order to show the effectiveness of observer (33), in Fig. 1 we have plotted the state x_2 in a solid line and its estimate in dashed line. The simulation is done for $k = 1$, $\alpha = 10$, and $\epsilon = 10^{-4}$. For $t \geq 11s$ the desired observation error $|x_2 - \hat{x}_2|$ is reached $(\simeq \epsilon)$, and φ is totally saturated. Suppose now that some additive controllers are present in the dynamics of the last reaction (30), i.e.,

$$\begin{cases} \dot{x}_1 = -k x_2 x_1^2 + u_1 \\ \dot{x}_2 = -k_d x_2^2 x_1 + u_2 \\ y = x_1 \end{cases} \tag{34}$$

then by elimination of the unmeasured state x_2 from the first equation of (34), we have $x_2 = u_1 - \dot{y}/ky^2 = u_1 - \dot{\widetilde{y}}(1 + y^2)/ky^2$. Consequently, the corresponding observer is

$$\begin{cases} \hat{x}_2 = \frac{u_1 - \xi_2(1+y^2)}{ky^2} \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = -\alpha^2 t^2 \left(\xi_1 - \arctan(y)\right) - 2\alpha t \xi_2. \end{cases} \tag{35}$$

IV. Other Schemes of Practical Observers

In this section, we show how can we combine the algebraic observer with classical Luenberger observer for nonlinear systems. Consider the nonlinear system

$$\begin{aligned} \dot{x} &= Ax + f(x, u) + g(y, u) \\ y &= Cx \end{aligned} \tag{36}$$

with the state x evolving on an open connected subset M of $\mathbb{R}^n$, the input $u \in \mathbb{R}^m$ and the output $y \in \mathbb{R}$; the vector valued $f\colon M \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is supposed to be smooth for simplicity with $f(0, 0) = 0$ and (A, C) is assumed to be an observable pair. The class of systems given in (36) is fairly general, but it is chosen herein for its popularity. If the state vector x verifies (27), then we rewrite the system dynamics (36) as follows:

$$\begin{aligned} \dot{x} &= Ax + \tilde{f}\left(\bar{x}, \bar{u}\right) + g(y, u) \\ y &= Cx \end{aligned} \tag{37}$$

where $\bar{x} = \left(y, \dot{\widetilde{y}}, \ldots, \widetilde{y}^{(\mu)}\right)$, $\widetilde{y} = \arctan(y)$, $\bar{u} = \left(u, \dot{u}, \ldots, u^{(\nu)}\right)$, $x = \psi(\bar{x}, \bar{u})$, and $\tilde{f}\left(\bar{x}, \bar{u}\right) = f(\cdot,\cdot) \circ \psi(\cdot,\cdot)$. The vector valued non-

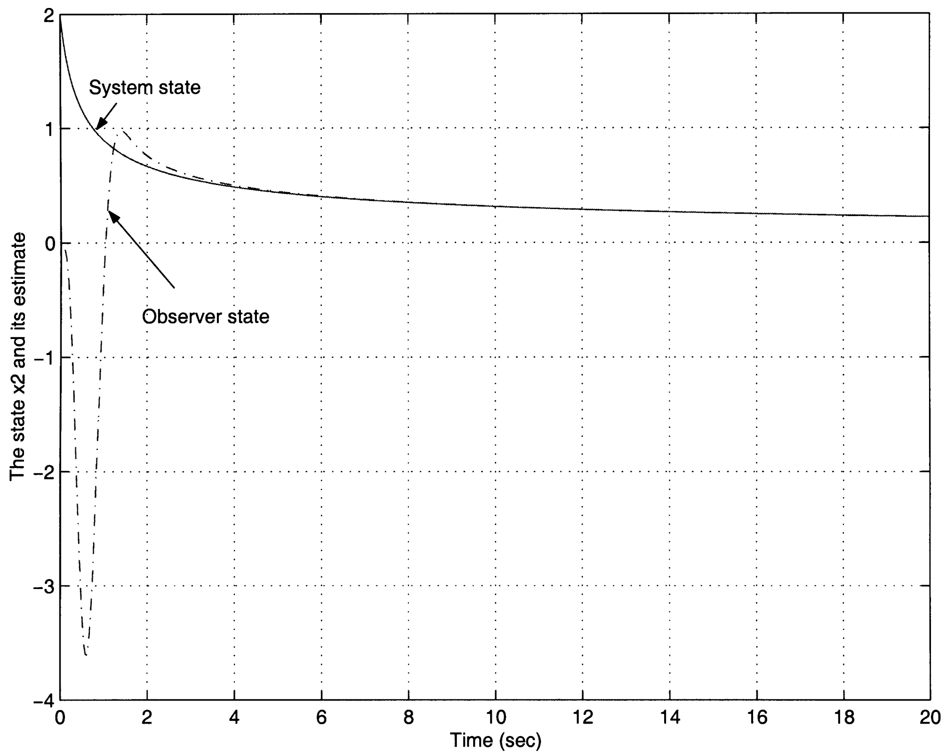

Fig. 1. State $\boldsymbol{x}_2$ and its estimate $\hat{\boldsymbol{x}}_2$.

linearity $\tilde{f}(\cdot,\cdot)\colon \mathbb{R}^{\mu+1} \times \mathbb{R}^{(\nu+1)m} \mapsto \mathbb{R}^n$ is supposed to be globally Lipschitz with respect to $\bar{x}$ with a Lipschitz constant λ, i.e., for all $\bar{x}_1$, $\bar{x}_2 \in \mathbb{R}^{\mu+1}$ and all $u \in U$

$$\left\|\tilde{f}(\bar{x}_1,\bar{u}) - \tilde{f}(\bar{x}_2,\bar{u})\right\| \leq \lambda \left\|\bar{x}_1 - \bar{x}_2\right\|. \tag{38}$$

The combination of the algebraic observer and the Luenberger observer is summarized in the following theorem.

Theorem 3: Consider system (37). For γ sufficiently large and for any $u \in U$ such that y is continuously differentiable, the following system:

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + \tilde{f}\left(\bar{\xi},\bar{u}\right) + g(y,u) + PC'\left(y - C\hat{x}\right)\\
\dot{\xi}_1 &= \xi_2\\
\dot{\xi}_2 &= -\alpha^2 t^2\left(\xi_1 - \arctan(y)\right) - 2\alpha t \xi_2\\
\dot{\xi}_i &= \xi_{i+1}\\
\dot{\xi}_{i+1} &= -\alpha^2 t^2\left(\xi_i - \xi_{i-1}\right) - 2\alpha t \xi_{i+1};\\
i &= 3,5,7,\ldots,2\mu-1\\
&AP + PA' - PC'CP + Q(\gamma) = 0
\end{aligned} \tag{39}$$

is an asymptotic converging observer of (37) where $Q(\gamma) = \operatorname{diag}\left[C_n^1\gamma^2, C_n^2\gamma^4,\ldots,C_n^n\gamma^{2n}\right]$, and $\bar{\xi}_i = \xi_{2i}, (1 \leq i \leq \mu)$.

Proof: Let $e = x - \hat{x}$, and let us take $V = e'P^{-1}e$ as a Lyapunov function to the error dynamics

$$\dot{e} = \left(A - PC'C\right)e + \tilde{f}\left(\bar{x},\bar{u}\right) - \tilde{f}\left(\bar{\xi},\bar{u}\right). \tag{40}$$

Then

$$\begin{aligned}
\dot{V} &= \dot{e}'P^{-1}e + e'P^{-1}\dot{e}\\
&= e'\left(A'P^{-1} + P^{-1}A - 2C'C\right)e\\
&\quad + 2e'P^{-1}\left(\tilde{f}\left(\bar{x},\bar{u}\right) - \tilde{f}\left(\bar{\xi},\bar{u}\right)\right).
\end{aligned} \tag{41}$$

Using $P^{-1}A + A'P^{-1} = C'C - P^{-1}Q(\gamma)P^{-1}$, then we obtain

$$\begin{aligned}
\dot{V} \leq & -e'\left(P^{-1}Q(\gamma)P^{-1} + C'C\right)e\\
& + 2e'P^{-1}\left(\tilde{f}\left(\bar{x},\bar{u}\right) - \tilde{f}\left(\bar{\xi},\bar{u}\right)\right)\\
\leq & -e'\left(P^{-1}Q(\gamma)P^{-1}\right)e\\
& + 2\left\|e'P^{-1}\right\|\left\|\tilde{f}\left(\bar{x},\bar{u}\right) - \tilde{f}\left(\bar{\xi},\bar{u}\right)\right\|.
\end{aligned} \tag{42}$$

Let $z = P^{-1}e$, then $V = z'Pz$. Using (38), we have

$$\begin{aligned}
\dot{V} &\leq -z'Q(\gamma)z + 2\lambda\|z\|\|\bar{\xi} - \bar{x}\|\\
&\leq -\lambda_{\min}\left(Q(\gamma)\right)\|z\|^2 + 2\lambda\|z\|\|\bar{\xi} - \bar{x}\|\\
&\leq -\frac{\gamma^2}{2}\|z\|^2 + 2\frac{\lambda^2}{\gamma^2}\|\bar{\xi} - \bar{x}\|^2\\
&\leq -\frac{\gamma^2}{2\lambda_{\max}(P)}V + 2\frac{\lambda^2}{\gamma^2}\|\bar{\xi} - \bar{x}\|^2.
\end{aligned} \tag{43}$$

Let $\mu = \gamma^2/2\lambda_{\max}(P)$, this gives $V \leq e^{-\mu t}V(0) + 2\lambda^2/\gamma^2 \int_0^t \left\|\bar{\xi}(s) - \bar{x}(s)\right\|^2 ds$, or

$$\|e\|^2 \leq C_1\|e(0)\|^2 e^{-\mu t} + C_2\int_0^t \|\bar{\xi}(s) - \bar{x}(s)\|^2 ds \tag{44}$$

such that $C_1 = \lambda_{\max}(P)/\lambda_{\min}(P)$, and $C_2 = 2\lambda^2/\gamma^2\lambda_{\min}(P)$. Let us take $\beta\left(\|e(0)\|^2, t\right) = C_1\|e(0)\|^2 e^{-\mu t}$, $r(t) = C_2\int_0^t \left\|\bar{\xi}(s) - \bar{x}(s)\right\|^2 ds$, then using the definition of input-to-state stability (ISS) and result of Theorem 1, then we conclude that the error dynamics is ISS stable with respect to the difference $\left\|\bar{\xi}(t) - \bar{x}(t)\right\|$; see [37] for more details on ISS.[1] Since all the estimates of the output derivatives converge asymptotically to the exact ones, then the observer error is asymptotically stable.

Remark 2: The sensitivity of observer (28) to noise is important, but observer (39) behaves so much resistant to eventual additive noise.

[1]System (23) is (globally) ISS if there exist a $\boldsymbol{KL}$-function $\boldsymbol{\beta}\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and a $\boldsymbol{K}$-function $\boldsymbol{r}$ such that for each $\boldsymbol{u}$ and $\boldsymbol{x}(0) \in \mathbb{R}^n$, it holds that $\|\boldsymbol{x}(t,\boldsymbol{x}(0),\boldsymbol{u})\| \leq \boldsymbol{\beta}(\|\boldsymbol{x}(0)\|, t) + \boldsymbol{r}(\|\boldsymbol{u}\|)$, for each $\boldsymbol{t} \geq 0$.

V. Conclusion

In this note, a new observer design methodology is presented. The whole design of the nonlinear observer is based upon differential algebraic concepts with time-varying linear system theory. We showed that the design method is free from several cumbersome computations and some strong geometric conditions, generally encountered in geometric observer design methods. The generalization of the observation procedure for multiple-input–multiple-output systems is possible and the observer strategy is exactly the same as we have developed for MISO systems. To do so, it is sufficient to recopy the dynamics of the MISO algebraic observer for different output signals. The simplicity and the straightforwardness of our observer design methodology give a first step for algebraic approach to nonlinear observer design.

Appendix

The Lipschitz Property of Uniformly Bounded Functions

When the output y is continuously differentiable and uniformly bounded, then using the definition of continuity, we could say that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $|y(t_1) - y(t_2)| < \epsilon$. For any $t_1 \neq t_2$, we can find $\eta > 0$ such that $|t_1 - t_2| > \eta$. Then, $|t_1 - t_2| < \delta$ implies $|y(t_1) - y(t_2)| < \epsilon/\eta \cdot \eta$. This gives $|y(t_1) - y(t_2)| < \epsilon/\eta |t_1 - t_2|$. When $|t_1 - t_2| \geq \delta$, we can write $|y(t_1) - y(t_2)| \leq 2\sup_{t\geq 0}|y(t)| = 2\sup_{t\geq 0}|y(t)|/\delta \cdot \delta \leq 2\sup_{t\geq 0}|y(t)|/\delta |t_1 - t_2|$. Finally, we conclude that for any $t_1 \neq t_2$, $|y(t_1) - y(t_2)| \leq \max\left\{2\sup_{t\geq 0}|y(t)|/\delta, \epsilon/\eta\right\}|t_1 - t_2|$. Consequently, $y(t)$ is globally Lipschitz.

Proof of Lemma 2

Here, the output $y(t)$ is assumed to be unbounded. We shall prove that the new output $\widetilde{y}(t) = \arctan(y(t))$ is uniformly bounded function. We have $\dot{\widetilde{y}}(t) = \dot{y}(t)/1 + y^2(t)$. Since $y(t) \in C^\infty$, then $\dot{\widetilde{y}}(t)$ is defined everywhere and $\lim_{t\to\infty}\int_0^t \dot{\widetilde{y}}(s)ds = -\arctan(y(0)) \pm \pi/2$. Using Barbalat's lemma, we conclude that $\lim_{t\to\infty}\dot{\widetilde{y}}(t) = 0$. With the same analysis and using the fact that the higher derivatives of $\widetilde{y}(t)$ are defined everywhere, we obtain

$$\lim_{t\to\infty}\int_0^t \widetilde{y}^{(i)}(s)ds = \lim_{t\to\infty}\left(\widetilde{y}^{(i-1)}(t) - \widetilde{y}^{(i-1)}(0)\right) = -\widetilde{y}^{(i-1)}(0), \qquad i \geq 2. \tag{45}$$

Then, $\lim_{t\to\infty}\widetilde{y}^{(i)}(t) = 0$ for $i \geq 1$, which implies that the derivatives $\widetilde{y}^{(i)}(t), i \geq 1$ are finite energy functions or uniformly bounded over $\mathbb{R}$.

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