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Static output feedback and guaranteed cost control of a class of discrete-time nonlinear systems with partial state measurements☆

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Abstract

In this paper both the static output feedback issue and the observer-based control of a class of discrete-time nonlinear systems are considered. Thanks to a newly developed linearization lemma, it is shown that the solution of the discrete-time output feedback problem is conditioned by a set of simple convex optimization conditions that are numerically tractable and free from any equality constraint. An illustrative example is provided to show the usefulness of the proposed control designs.

Keywords: Nonlinear observers; Discrete-time systems; System theory; Linear matrix inequalities; Robust control

1. Introduction

Uncertainty is an inherent problem in modelling of practical processes. The environment in which a process operates also introduces uncertainties due to disturbances and inability to measure properly the internal physical states or parameters. Thus, the handling of uncertainty is an integral feature of any robust control design. Basically, the main origin of uncertainty in physical systems appears from the inability to perfectly characterize the evolution of a dynamical system as a deterministic set of differential equations. As consequence, no mathematical model can precisely reflect the true dynamics of a physical system; moreover, the mismatch between a mathematical model and a physical system can be extremely large. Uncertain dynamical systems consisting of dynamical plants having uncertain parameters or unmodelled dynamics have been extensively studied in the literature; see e.g., [16,12,1] and the references therein.

Since the state vector is not often available for feedback, the estimation of the unmeasured states constitutes a crucial task for any stabilization process. State reconstruction and control of discrete-time nonlinear systems have been the subject of numerous research papers; see e.g., [20,19,24,11,2,15,5] and the references therein. However, the static and dynamic output feedback control of discrete-time nonlinear systems has received little attention. When the state vector of the considered system is not completely available for feedback, both the static and the dynamic output feedback solutions can be considered. The static output feedback problem is known to be a challenging issue due to

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its non-convex nature and many attempts have been made to solve the linear case problem; see for example [1,21,13, 4,7,6]. When the system is not stabilizable by a static output feedback, the dynamic output feedback solution remains the only possible way to achieve the desired aim. However, in such a control problem the coupling of variables to be determined (generally characterized as the controller and the observer gains) makes the observer-based issue a non-convex one.

In this paper we give a new systematic approach to the static and dynamic output feedback for a class of discrete-time nonlinear systems subject to external norm-bounded uncertainties. We assume that the nonlinear system can be stabilized by either static output or linear full state feedbacks. It is important to outline that several attempts to stabilization of nonlinear systems by linear feedbacks have appeared; see e.g., [23,1]. However, little constructive design methodology has been proposed for the stabilization of discrete-time nonlinear systems with partial state measurements. In order to cope with these inherent issues, first, a new *linearization lemma* is developed. Thanks to this result, both the static and the dynamic output feedback problems are completely solved, in an efficiently numerically way and without any equality constraint. Subsequently, sufficient linear matrix inequality conditions are given for ensuring the stability of the discrete-time nonlinear system under the action of a nonlinear observer-based controller with uncertainty attenuation. The power of the developed results is demonstrated through a numerical example. Throughout this paper, the notation $A > 0$ (respectively $A < 0$) means that the matrix A is positive definite (respectively negative definite). We denote by A^{T} the matrix transpose of A. We denote by I and $\mathbf{0}$ the identity matrix and the null matrix of appropriate dimensions, respectively. $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{Z}$ stand for the sets of natural, real and integer numbers, respectively. $\|\cdot\|$ stands for the usual Euclidean norm. "$\star$" is used to indicate a matrix element that is induced by transposition.

2. Static output feedback

Consider the discrete-time nonlinear system

$$\begin{cases} x_{k+1} = Ax_k + f(x_k) + Bu_k + D^{(\xi)}\xi_k; \quad k \in \mathbb{Z}_{\geq 0}, \\ y_k = Cx_k, \end{cases} \tag{1}$$

where the nominal matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D^{(\xi)} \in \mathbb{R}^{n\times n}$ are constant real matrices. The norm-bounded vector $\xi_k \in \mathbb{R}^n$ represents the system uncertainty. We assume that the pair (A, C) is detectable and the pair (A, B) is controllable. $x_k \in \mathscr{M} \subseteq \mathbb{R}^n$, $u_k \in \mathscr{U}$ is an m-dimensional control input, $\mathscr{U}$ is the set of inputs for which system (1) is observable. $y_k \in \mathbb{R}^p$ is the system output, and $f : \mathscr{M} \to \mathbb{R}^n$ is a $\mathscr{C}^1$ Lipschitz nonlinearity satisfying $f(0) = 0$ and

$$\frac{\partial f(x_k)}{\partial x_k} \in \mathbf{Co}\{F_1, F_2, \ldots, F_\nu\}, \tag{2}$$

where the symbol "**Co**" stands for the convex hull and $(F_i)_{1\leq i\leq \nu}$ are the associated convex hull matrices. We say that the Jacobian $\frac{\partial f(x_k)}{\partial x_k}$ belongs to a convex polytopic set defined as

$$\mathscr{F} = \left\{ \mathscr{F}(\beta) = \sum_{i=1}^{\nu} \beta_i F_i, \sum_{i=1}^{\nu} \beta_i = 1, \beta_i \geq 0 \right\}. \tag{3}$$

Remark 1. The global Lipschitz condition of the system nonlinearity is rather restrictive and is not usually verified due to the inherently nonlinear dynamics of existing physical systems. But there are many practical systems that verify this condition; see for example [18]. In observer design the condition of Lipschitz plays a fundamental role in inducing the desired estimation error dynamics endowed with the requisite stability characteristic and it can therefore be made satisfied in well-defined state space regions. Successfully examples of observation and control schemes under the Lipschitz condition are numerous and the reader is referred to the references included in this paper for further details.

Our first objective is to give sufficient linear matrix inequality conditions for stabilization of system (1) by a static output controller $u_k = Ly_k$. Notice that the static output feedback for linear systems is a challenging issue and many contributions have been devoted to this crucial problem. However, this particular issue remains an open problem since

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sufficient and necessary conditions for the existence of such stabilizing output static feedback are not numerically tractable due to the resulting non-convex optimization problem; see, e.g., [13,3]. Recently, a linear matrix inequality approach has been proposed for solving the static output feedback problem in the continuous-time and the discrete-time cases; see [4,6]. In Reference [4] the proposed LMI is subject to an equality constraint that permits one implicitly to reverse the non-convex optimization issue to a convex one. The reader is also referred to the references [17,8,10,22,4] for more challenging results. For more details on the static output feedback, the reader is referred to the survey paper [21].

In this section, we propose sufficient linear matrix inequality conditions for solving the static output feedback problem without any equality constraint. Further, the result concerns nonlinear discrete-time systems with Lipschitzian nonlinearities, and can be applied to discrete-time linear systems with multi-objective stabilization issues. Before presenting the main results of this paper, we introduce the following key lemma which will be used in setting the proofs of the next statements.

Lemma 1. *Given the matrices X, Y, and Z of appropriate dimensions where $X = X^{\mathrm{T}} > 0$, and $Z = Z^{\mathrm{T}} > 0$, then, the following linear matrix inequality holds:*

$$\begin{bmatrix} -X & Y^{\mathrm{T}} \\ Y & -Z^{-1} \end{bmatrix} < 0, \tag{4}$$

if there exists a positive constant α such that

$$\begin{bmatrix} -X & \alpha Y^{\mathrm{T}} & \mathbf{0} \\ \alpha Y & -2\alpha \boldsymbol{I} & Z \\ \mathbf{0} & Z & -Z \end{bmatrix} < 0. \tag{5}$$

Proof. In order to prove the sufficiency of (5), one could easily verify that

$$\begin{bmatrix} -X & Y^{\mathrm{T}} \\ Y & -Z^{-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dfrac{\boldsymbol{I}}{\alpha} & Z^{-1} \end{bmatrix} \begin{bmatrix} -X & \alpha Y^{\mathrm{T}} & \mathbf{0} \\ \alpha Y & -2\alpha \boldsymbol{I} & Z \\ \mathbf{0} & Z & -Z \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \dfrac{\boldsymbol{I}}{\alpha} \\ \mathbf{0} & Z^{-1} \end{bmatrix}. \tag{6}$$

This ends the proof. □

Remark that, if we start from inequality (4), we can say that (4) is satisfied if and only if there exists $\alpha > 0$ such that $-X + (\alpha Y)^{\mathrm{T}} \frac{Z}{\alpha^2} (\alpha Y) < 0$, which is equivalent by the Schur complement to the following LMI:

$$\begin{bmatrix} -X & \alpha Y^{\mathrm{T}} \\ \alpha Y & -\alpha^2 Z^{-1} \end{bmatrix} < 0. \tag{7}$$

Using the fact that for any $\alpha > 0$ and $Z = Z^{\mathrm{T}} > 0$, we have $2\alpha \boldsymbol{I} \leq \alpha^2 Z^{-1} + Z$. If

$$\begin{bmatrix} -X & \alpha Y^{\mathrm{T}} \\ \alpha Y & -2\alpha \boldsymbol{I} + Z \end{bmatrix} < 0, \tag{8}$$

then (4) is satisfied. It is clear that inequality (8), which is equivalent by the Schur complement to (5), becomes a necessary condition for some $Z > 0$. Further, inequality (5) does not restrict the domain of the search of $Z > 0$ which is $0 < Z < 2\alpha \boldsymbol{I}$. In other words, the upper bound of Z is parameterized by α which is one of the LMI variables. Since $-2\alpha \boldsymbol{I} + Z$ appears in the diagonal of the matrix inequality (8), then condition $Z < 2\alpha \boldsymbol{I}$ must be satisfied to fulfill inequality (8). We shall herein call Lemma 1 the *linearization lemma* since inequality (5) is linear with respect to X, Z, and α if Y is known. If we put $\widetilde{Y} = \alpha Y$, then inequality (8) can be solved with respect to X, $\widetilde{Y}$, Z, and α.

Theorem 1. *System (1) satisfying (3) with $\xi_k \equiv 0$ is globally asymptotically stable under the action of the static output feedback $u_k = \frac{\hat{L}}{\alpha} y_k$ provided that there exist a positive definite matrix $X = X^{\mathrm{T}} \in \mathbb{R}^{n \times n}$, a real matrix $\hat{L} \in \mathbb{R}^{m \times p}$,* *and a positive constant α such that the following linear matrix inequalities hold:*

$$\begin{bmatrix} -X & \alpha(A^{\mathrm{T}}+F_i^{\mathrm{T}})+C^{\mathrm{T}}\hat{L}^{\mathrm{T}}B^{\mathrm{T}} & \mathbf{0} \\ \alpha(A+F_i)+B\hat{L}C & -2\alpha\boldsymbol{I} & X \\ \mathbf{0} & X & -X \end{bmatrix} < 0, \quad 1 \le i \le \nu, \tag{9}$$

where ν is the number of the convex hull matrices of the Jacobian of $f(x_k)$.

Proof. Consider the discrete-time nonlinear system (1) under the action of the feedback $u_k = Ly_k$ with $\xi_k \equiv 0$. Then, by the use of the mean-value theorem, the closed-loop dynamics can be written as

$$x_{k+1} = (A+BLC)x_k + \int_0^1 \left.\frac{\partial f(\tau_k)}{\partial \tau_k}\right|_{\tau_k=(1-s)x_k} x_k \mathrm{d}s. \tag{10}$$

Taking $V_k = x_k^{\mathrm{T}} X x_k$ as a Lyapunov function, we have

$$\begin{aligned} V_{k+1} - V_k = {} & \left[(A+BLC)x_k + \int_0^1 \left.\frac{\partial f(\tau_k)}{\partial \tau_k}\right|_{\tau_k=(1-s)x_k} x_k \mathrm{d}s\right]^{\mathrm{T}} X \\ & \times \left[(A+BLC)x_k + \int_0^1 \left.\frac{\partial f(\tau_k)}{\partial \tau_k}\right|_{\tau_k=(1-s)x_k} x_k \mathrm{d}s\right] - x_k^{\mathrm{T}} X x_k. \end{aligned} \tag{11}$$

Since for any constant symmetric matrix $M \in \mathbb{R}^{n\times n}$, $M = M^{\mathrm{T}} > 0$, scalar $\gamma > 0$, vector function $\omega : [0,\gamma] \mapsto \mathbb{R}^n$ such that the integration in the following is well defined, we have

$$\gamma \int_0^\gamma \omega^{\mathrm{T}}(\beta) M \omega(\beta) \mathrm{d}\beta \ge \left(\int_0^\gamma \omega(\beta)\mathrm{d}\beta\right)^{\mathrm{T}} M \left(\int_0^\gamma \omega(\beta)\mathrm{d}\beta\right) \tag{12}$$

(see [9] for the proof), then we can write that $V_{k+1} - V_k < 0$ if the following holds:

$$\int_0^1 \left\{ \left[(A+BLC) + \left.\frac{\partial f(\tau_k)}{\partial \tau_k}\right|_{\tau_k=(1-s)x_k}\right]^{\mathrm{T}} X \times \left[(A+BLC) + \left.\frac{\partial f(\tau_k)}{\partial \tau_k}\right|_{\tau_k=(1-s)x_k}\right] - X \right\} \mathrm{d}s < 0. \tag{13}$$

which is equivalent by the Schur complement lemma to the following matrix inequality:

$$\int_0^1 \begin{bmatrix} -X & A^{\mathrm{T}} + C^{\mathrm{T}}L^{\mathrm{T}}B^{\mathrm{T}} + \left.\left(\dfrac{\partial f(\tau_k)}{\partial \tau_k}\right)^{\mathrm{T}}\right|_{\tau_k=(1-s)x_k} \\ \star & -X^{-1} \end{bmatrix} \mathrm{d}s < 0. \tag{14}$$

Since $\frac{\partial f(\tau_k)}{\partial \tau_k}$ is norm-bounded and continuous for all $k \in \mathbb{Z}_{\ge 0}$ and all the matrices involved in (14) are real, then, the integration in (14) is well defined. Using the fact that

$$\begin{aligned} & \begin{bmatrix} -X & A^{\mathrm{T}} + C^{\mathrm{T}}L^{\mathrm{T}}B^{\mathrm{T}} + \left.\left(\dfrac{\partial f(\tau_k)}{\partial \tau_k}\right)^{\mathrm{T}}\right|_{\tau_k=(1-s)x_k} \\ \star & -X^{-1} \end{bmatrix} \\ & \quad \in \mathbf{Co}\left\{ \begin{bmatrix} -X & A^{\mathrm{T}} + F_i^{\mathrm{T}} + C^{\mathrm{T}}L^{\mathrm{T}}B^{\mathrm{T}} \\ \star & -X^{-1} \end{bmatrix}, 1 \le i \le \nu \right\} \end{aligned} \tag{15}$$

then, after replacing the Jacobian matrix by its convex hull matrices, sufficient conditions for fulfilling (14) are

$$\begin{bmatrix} -X & A^{\mathrm{T}} + F_i^{\mathrm{T}} + C^{\mathrm{T}}L^{\mathrm{T}}B^{\mathrm{T}} \\ \star & -X^{-1} \end{bmatrix} < 0, \quad 1 \le i \le \nu. \tag{16}$$

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If we make the replacement

$$\begin{aligned} &A + F_i + BLC = Y_i; \quad 1 \le i \le \nu, \\ &Z = X, \end{aligned} \tag{17}$$

then, from result of Lemma 1, we can say that

$$\begin{bmatrix} -X & Y_i^{\mathrm{T}} \\ \star & -Z^{-1} \end{bmatrix} < 0, \quad 1 \le i \le \nu \tag{18}$$

are satisfied if conditions (9) are verified. This ends the proof. □

3. Guaranteed cost control via static output feedback

In order to show the usefulness of the linearization lemma, let us consider the optimal control problem of system (1) under the feedback $u_k = Ly_k$ and $\xi_k \equiv 0$. The objective is to find the gain L such that for all initial conditions $x_0 \in \mathbb{R}^n$

$$\mathscr{J}_\infty = \sum_{k=0}^{\infty} x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k < x_0^{\mathrm{T}} P x_0, \tag{19}$$

where $Q = Q^{\mathrm{T}} \ge 0$ and $R = R^{\mathrm{T}} > 0$ are some prescribed real matrices and $P = P^{\mathrm{T}}$ is a positive definite matrix to be determined. If we put $V_k = x_k^{\mathrm{T}} P x_k$ as a Lyapunov function, and $A_L(i) = A + F_i + BLC$, then for a given natural number N, we have

$$\begin{aligned} \mathscr{J}_N &= \sum_{k=0}^{N} \left\{ x_k^{\mathrm{T}} Q x_k + x_k^{\mathrm{T}} C^{\mathrm{T}} L^{\mathrm{T}} R L C x_k + V_{k+1} - V_k \right\} - V_{N+1} \\ &\le \sum_{k=0}^{N} \left\{ x_k^{\mathrm{T}} Q x_k + x_k^{\mathrm{T}} C^{\mathrm{T}} L^{\mathrm{T}} R L C x_k + V_{k+1} - V_k \right\} \\ &\le \sum_{k=0}^{\infty} \left\{ x_k^{\mathrm{T}} (Q + C^{\mathrm{T}} L^{\mathrm{T}} R L C + A_L^{\mathrm{T}}(i) P A_L(i) - P) x_k \right\}. \end{aligned}$$

This implies that if

$$Q + C^{\mathrm{T}} L^{\mathrm{T}} R L C + A_L^{\mathrm{T}}(i) P A_L(i) - P < 0, \quad \forall i \tag{20}$$

is verified then the static output feedback $u_k = L\, y_k$ minimizes the criterion (19). The last matrix inequality is equivalent by the Schur complement lemma to the following matrix inequality:

$$\begin{bmatrix} -P + Q & C^{\mathrm{T}} L^{\mathrm{T}} & A_L^{\mathrm{T}}(i) \\ LC & -R^{-1} & \mathbf{0} \\ A_L(i) & \mathbf{0} & -P^{-1} \end{bmatrix} < 0. \tag{21}$$

Let us define $X = -P + Q$, $Y_i = \begin{bmatrix} LC \\ A_L(i) \end{bmatrix}$; $1 \le i \le \nu$, $Z = \begin{bmatrix} R & \mathbf{0} \\ \star & P \end{bmatrix}$; then using the result of Lemma 1, we can conclude that (21) holds if there exists $\alpha > 0$ and $\hat{L}$ with appropriate dimension such that the following LMIs hold for all $1 \le i \le \nu$:

$$\begin{bmatrix} -P + Q & \star & \star \\ \hat{L} C & -2\alpha I + R & \star \\ A_L^{(\alpha)}(i) & \mathbf{0} & -2\alpha I + P \end{bmatrix} < 0, \tag{22}$$

where $A_L^{(\alpha)}(i) = \alpha(A + F_i) + B\hat{L}C$. This result is summarized in the following statement.

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Theorem 2. *Consider the discrete-time nonlinear system (1) with $\xi_k \equiv 0$ and let $Q \geq 0$ and $R = R^{\mathrm{T}} > 0$ be two given symmetric real matrices. If there exists $\alpha > 0$, a symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$, and a matrix $\hat{L} \in \mathbb{R}^{m\times p}$ such that the linear matrix inequalities (22) hold for $1 \leq i \leq \nu$ then, system (1) under the above conditions is asymptotically stable under the static output feedback $u_k = \frac{1}{\alpha}\hat{L}y_k$ and $\sum_{k=0}^{\infty} x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k < x_0^{\mathrm{T}} P x_0$, for all initial conditions $x_0 \in \mathbb{R}^n$.*

4. Observer-based guaranteed cost control

In this section we tackle the stabilization issue under the assumption that the full state vector is not available for feedback. This unavoidable problem that is encountered in any nonlinear observer-based control strategy brings us to studying, in LMI framework, the *separation principle* for this class of nonlinear discrete-time systems. That is to say, stability of nonlinear systems under the action of an observer-based linear controller is not always guaranteed, and hence, the stability of the observation error along with the stability of the system states must be checked simultaneously. To this end, let us consider system (1) and its corresponding state observer described by the following dynamics:

$$\hat{x}_{k+1} = A\hat{x}_k + f(\hat{x}_k) + Bu_k + L(C\hat{x}_k - Cx_k), \tag{23}$$

where L is the observer gain to be determined. Under the action of the observer-based feedback $u_k = K\hat{x}_k$, the closed-loop dynamics is given by

$$\begin{cases} x_{k+1} = Ax_k + f(x_k) + Bu_k + D^{(\xi)}\xi_k, \\ \hat{x}_{k+1} = A\hat{x}_k + f(\hat{x}_k) + Bu_k + L(C\hat{x}_k - y_k), \\ u_k = K\hat{x}_k. \end{cases} \tag{24}$$

Let $e_k = \hat{x}_k - x_k$ be the observation error. Then, under the feedback $u_k = K\hat{x}_k$, we write

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \int_0^1 \varphi(x_k, \hat{x}_k, e_k, s) \begin{bmatrix} x_k \\ e_k \end{bmatrix} \mathrm{d}s + D\xi_k^{(\theta)}, \tag{25}$$

where

$$\varphi(x_k, \hat{x}_k, e_k, s) = \begin{bmatrix} A + BK + \left.\dfrac{\partial f(\tau_k)}{\partial \tau_k}\right|_{\tau_k=(1-s)x_k} & BK \\ \mathbf{0} & A + LC + \left.\dfrac{\partial f(\tau_k)}{\partial \tau_k}\right|_{\tau_k=\hat{x}_k - se_k} \end{bmatrix},$$

$$D = \begin{bmatrix} D^{(\xi)} & \mathbf{0} \\ \mathbf{0} & -D^{(\xi)} \end{bmatrix}, \qquad \xi_k^{(\theta)} = \begin{bmatrix} \xi_k \\ \xi_k \end{bmatrix}. \tag{26}$$

The controller and the observer gains are chosen such that system (25) is asymptotically stable when $\xi_k^{(\theta)} \equiv 0$, and for the zero initial conditions $\|C_\varphi \theta_k\| \leq \gamma \|\xi_k^{(\theta)}\| \forall k$ where $C_\varphi = \begin{bmatrix} C & \mathbf{0} \end{bmatrix}$, $\gamma > 0$ and $\theta_k = \begin{bmatrix} x_k & e_k \end{bmatrix}^{\mathrm{T}}$. Define $W_k = \theta_k^{\mathrm{T}} P \theta_k$, where $P \in \mathbb{R}^{2n\times 2n}$ is a symmetric and positive definite matrix, and let $\mathscr{R}_N = \sum_{k=0}^{N} \left\{ \theta_k^{\mathrm{T}} C_\varphi^{\mathrm{T}} C_\varphi \theta_k - \gamma^2 \xi_k^{(\theta)^{\mathrm{T}}} \xi_k^{(\theta)} \right\}$, $N \in \mathbb{N}_{>0}$. Then, $\mathscr{R}_N$ can be rewritten as

$$\begin{aligned} \mathscr{R}_N &= \sum_{k=0}^{N} \left\{ \theta_k^{\mathrm{T}} C_\varphi^{\mathrm{T}} C_\varphi \theta_k - \gamma^2 \xi_k^{(\theta)^{\mathrm{T}}} \xi_k^{(\theta)} + W_{k+1} - W_k \right\} - W_{N+1} \\ &\leq \sum_{k=0}^{N} \left\{ \theta_k^{\mathrm{T}} C_\varphi^{\mathrm{T}} C_\varphi \theta_k - \gamma^2 \xi_k^{(\theta)^{\mathrm{T}}} \xi_k^{(\theta)} + W_{k+1} - W_k \right\} \\ &= \sum_{k=0}^{N} \left\{ \theta_k^{\mathrm{T}} C_\varphi^{\mathrm{T}} C_\varphi \theta_k - \gamma^2 \xi_k^{(\theta)^{\mathrm{T}}} \xi_k^{(\theta)} + \left(\int_0^1 \xi_k^{(\theta)^{\mathrm{T}}} D^{\mathrm{T}} + \theta_k^{\mathrm{T}} \varphi^{\mathrm{T}}(x_k, \hat{x}_k, e_k, s) \mathrm{d}s \right) P \right. \end{aligned}$$

$$\times \left(\int_0^1 D\xi_k^{(\theta)} + \varphi(x_k,\ \hat{x}_k,\ e_k,\ s)\theta_k \mathrm{d}s \right) - \int_0^1 \theta_k^{\mathrm{T}} P \theta_k \mathrm{d}s \right\}. \tag{27}$$

We have

$$\left(\int_0^1 \xi_k^{(\theta)^{\mathrm{T}}} D^{\mathrm{T}} + \theta_k^{\mathrm{T}} \varphi^{\mathrm{T}}(x_k, \hat{x}_k, e_k, s)\mathrm{d}s \right) P \left(\int_0^1 D\xi_k^{(\theta)} + \varphi(x_k, \hat{x}_k, e_k, s)\theta_k \mathrm{d}s \right)$$
$$\le \int_0^1 \left(\xi_k^{(\theta)^{\mathrm{T}}} D^{\mathrm{T}} + \theta_k^{\mathrm{T}} \varphi^{\mathrm{T}}(x_k, \hat{x}_k, e_k, s) \right) P \left(D\xi_k^{(\theta)} + \varphi(x_k, \hat{x}_k, e_k, s)\theta_k \right) \mathrm{d}s \tag{28}$$

This implies that

$$\mathscr{R}_N \le \sum_{k=0}^{\infty} \int_0^1 \begin{bmatrix} \theta_k \\ \xi_k^{(\theta)} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \varphi^{\mathrm{T}} P \varphi - P + C_\varphi^{\mathrm{T}} C_\varphi & \varphi^{\mathrm{T}} P D \\ D^{\mathrm{T}} P \varphi & D^{\mathrm{T}} P D - \gamma^2 \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \theta_k \\ \xi_k^{(\theta)} \end{bmatrix} (x_k, \hat{x}_k, e_k, s)\mathrm{d}s. \tag{29}$$

By the Schur complement, $\mathscr{R}_N < 0$ if

$$\int_0^1 \begin{bmatrix} -P + C_\varphi^{\mathrm{T}} C_\varphi & \mathbf{0} & \varphi^{\mathrm{T}} \\ \mathbf{0} & -\gamma^2 \boldsymbol{I} & D^{\mathrm{T}} \\ \varphi & D & -P^{-1} \end{bmatrix} (x_k, \hat{x}_k, e_k, s)\mathrm{d}s < 0. \tag{30}$$

Since all the elements of the matrix

$$\begin{bmatrix} -P + C_\varphi^{\mathrm{T}} C_\varphi & \mathbf{0} & \varphi^{\mathrm{T}} \\ \mathbf{0} & -\gamma^2 \boldsymbol{I} & D^{\mathrm{T}} \\ \varphi & D & -P^{-1} \end{bmatrix} (x_k, \hat{x}_k, e_k, s) \tag{31}$$

are well defined and continuous, then, (30) is satisfied if the matrix inequality

$$\begin{bmatrix} -P + C_\varphi^{\mathrm{T}} C_\varphi & \mathbf{0} & \varphi^{\mathrm{T}} \\ \mathbf{0} & -\gamma^2 \boldsymbol{I} & D^{\mathrm{T}} \\ \varphi & D & -P^{-1} \end{bmatrix} (x_k, \hat{x}_k, e_k, s) < 0 \tag{32}$$

is always satisfied. Putting

$$X = \begin{bmatrix} -P + C_\varphi^{\mathrm{T}} C_\varphi & \mathbf{0} \\ \mathbf{0} & -\gamma^2 \boldsymbol{I} \end{bmatrix}, \qquad Y = \begin{bmatrix} \varphi & D \end{bmatrix}, \qquad Z = P, \tag{33}$$

then, by the use of result of Lemma 1, we can say that (32) is satisfied if

$$\begin{bmatrix} -P + C_\varphi^{\mathrm{T}} C_\varphi & \mathbf{0} & \alpha\varphi^{\mathrm{T}} \\ \mathbf{0} & -\gamma^2 \boldsymbol{I} & \alpha D^{\mathrm{T}} \\ \alpha\varphi & \alpha D & -2\alpha \boldsymbol{I} + P \end{bmatrix} (x_k, \hat{x}_k, e_k, s) < 0. \tag{34}$$

By replacing the Jacobians that appear in the matrix $\varphi(\cdot)$ by their convex hull matrices, and setting $K = \widetilde{K}/\alpha$, and $L = \widetilde{L}/\alpha$, then, we can write that inequality (30) holds if the following linear matrix inequalities hold for $1 \le i,\ j \le \nu$:

$$\begin{bmatrix} -P + C_\varphi^{\mathrm{T}} C_\varphi & \mathbf{0} & \begin{bmatrix} \alpha(A + F_i) + B\widetilde{K} & B\widetilde{K} \\ \mathbf{0} & \alpha(A + F_j) + \widetilde{L}C \end{bmatrix}^{\mathrm{T}} & \mathbf{0} \\ \star & -\gamma^2 \boldsymbol{I} & \alpha D^{\mathrm{T}} & \mathbf{0} \\ \star & \star & -2\alpha \boldsymbol{I} & P \\ \star & \star & \star & -P \end{bmatrix} < 0. \tag{35}$$

We have proved the following statement.

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Theorem 3. *Consider the closed-loop system (24) with* $K = \frac{\widetilde{K}}{\alpha}$, *and* $L = \frac{\widetilde{L}}{\alpha}$. *Provided that there exist a symmetric and positive definite matrix* $P \in \mathbb{R}^{2n\times 2n}$, *a real matrix* $\widetilde{K} \in \mathbb{R}^{m\times n}$, *a real matrix* $\widetilde{L} \in \mathbb{R}^{n\times p}$, *and two positive constants* α *and* γ *such that the linear matrix inequalities (35) hold for* $1 \le i, j \le \nu$, *then, system (24) under the feedback* $u_k = \frac{\widetilde{K}}{\alpha}\hat{x}_k$ *is globally asymptotically stable when* $\xi_k = 0$, *and satisfies the optimality condition* $\|C_\varphi \theta_k\| \le \gamma \|\xi_k^{(\theta)}\|$ *for the zero initial conditions and non-zero* $\xi_k \in \mathscr{L}_2[0\ \infty)$.

To illustrate the observer-based results, consider the discrete-time nonlinear system

$$x_{k+1} = \begin{bmatrix} \frac{1}{3} & \frac{3}{2} & 0 \\ \frac{1}{2} & \frac{3}{5} & 1 \\ 1 & -1 & \frac{1}{3} \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \frac{1}{7}\dfrac{x_k^{(1)}}{1+x_k^{(1)2}} \\ 0 \end{bmatrix} + 0.1 \begin{bmatrix} \xi_k^{(1)} \\ \xi_k^{(2)} \\ \xi_k^{(3)} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} u_k,$$
$$y_k = \begin{bmatrix} -\frac{1}{9} & \frac{1}{2} & 0 \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{9} \end{bmatrix} x_k. \tag{36}$$

One can easily verify that the vertices of the Jacobian of the system nonlinearity are $F_1 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{56} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $F_2 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{7} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. After solving the LMIs (35) with respect to their variables, we find

$$P = \begin{bmatrix} 1.9987 & -1.2785 & 0.6196 & 0.0726 & -0.6200 & -0.0891 \\ -1.2785 & 2.3167 & -0.4047 & -0.1264 & -0.4452 & -0.1597 \\ 0.6196 & -0.4047 & 0.5717 & -0.0091 & -0.1010 & -0.2169 \\ 0.0726 & -0.1264 & -0.0091 & 3.3229 & 0.4944 & -0.1226 \\ -0.6200 & -0.4452 & -0.1010 & 0.4944 & 3.1674 & -0.0638 \\ -0.0891 & -0.1597 & -0.2169 & -0.1226 & -0.0638 & 4.1505 \end{bmatrix},$$
$$L = \frac{1}{\alpha}\begin{bmatrix} -5.3718 & -4.9948 \\ 0.5224 & -14.0928 \\ 7.4701 & -8.4182 \end{bmatrix}, \quad K = \frac{1}{\alpha}\begin{bmatrix} -2.5489 & -3.8793 & -5.9568 \\ -1.2820 & -5.1116 & -0.0327 \end{bmatrix},$$
$$\alpha = 2.3779, \gamma = 0.7934. \tag{37}$$

The sufficient conditions given in the statements of Theorems 1–3 lead to a simple and straightforward design of static and dynamic output controllers for a class of discrete-time nonlinear systems. The main feature of the proposed LMI-based design is that the formulation of the observer and the controller gains is given in terms of a set of linear matrix inequalities that must hold simultaneously. In contrast to the design proposed for linear continuous-time systems [14], the solvability of LMIs (9), (22) and (35) is not iterative and does not involve equality constraints. Therefore, the implementation of the proposed LMI-based procedures becomes an extremely easy task. It is worth mentioning that the design is not restricted to some class of triplets (A, B, C). However, these conditions are just sufficient and may be sensitive to state transformations.

5. Conclusion

In this paper, new sufficient linear matrix inequality conditions for static output feedback stabilization of discrete-time Lipschitzian nonlinear systems are presented. Subsequently, sufficient LMI conditions for guaranteed cost control by static output feedback are given. Observer-based control with uncertainty attenuation is also discussed in the convex optimization setting without any equality constraint. Thanks to the proposed linearization lemma, the LMIs developed for both the static and the dynamic output feedbacks are not subject to any equality constraint and are easily solved by any LMI software. Note that the result of the linearization lemma is not limited to the simplification of the static and the dynamic output feedback issues, but it has other implication in handling various multiple-objective control designs in the LMI setting.

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