A NUMERICAL OBSERVER CONTROLLER FOR THE STABILIZATION OF AN INVERTED PENDULUM SYSTEM.

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Abstract. This paper deals with the design of two independent nonlinear controllers that swing-up and stabilize an inverted pendulum. The control strategy is established without any approximation of the nonlinear terms in the model equations. The observer design is achieved through the so-called regularization techniques by B-spline functions. Numerical simulations are presented to illustrate the effectiveness of the control procedure.

Keywords: Nonlinear observers, Numerical methods, Nonlinear control, Pendulum.

1. INTRODUCTION

Historically, nonminimum phase underactuated systems, i.e, systems where the number of control inputs is less than the number of degrees of freedom, have been given a lot of attention and their properties have been studied in detail. Some of these systems behave as linear systems after an invertible nonlinear change of variables and have been classified with the aid of methods from differential geometry. Other systems were studied in the context of differential flatness theory which gives powerful tools for trajectory generation for nonlinear systems [11], [17]. However, the control of certain systems remains challenging, because their properties cannot verified or because they do not satisfy the linearization conditions. The inverted pendulum is one of those systems which is partially linearizable and its control remains an open problem, see [21], [12], [15] for recent works in this domain.

Although controllers of very different type exist for the inverted pendulum, all these controllers have in common that they are discontinuous. Moreover, the observation of the unobservable states, generally the velocities, necessitates a suitable change of coordinates to put the system in its canonical form. In this article we shall develop two discontinuous nonlinear controllers: the first is conceived to make the stable equilibrium point of the pendulum an unstable point until the pendulum leaves the position where the controller is singular. The second will then catch the pendulum and steer the whole system to the desired equilibrium. A part of this article is devoted to the presentation of a numerical computational method used as an obsever. It can be proved that the controller-observer converges if the approximation error of the numerical method are bounded, but the proof will not be written out here fore sake of brevity.

The paper is organized as follows. In Section 2, the state model describing the dynamics of the inverted pendulum is presented. Section 3 is devoted to the construction of a nonlinear controller that swings up the pendulum to its unstable equilibrium without concern of the cart displacement and the design of another controller which steers the cart and the pendulum to their final equilibriums. The numerical method of velocity observation is presented in Section 4. In Section 5 simulation results are given to demonstrate the effectiveness of the control strategy.

2. THE MODEL

Consider the inverted pendulum illustrated in Fig. 1. A cart of mass M_0 has a composite pendulum (stick of mass M_1, and another mass M attached in the end of the stick). The inertia momentums of the stick and the mass M with respect to their gravity centers are J_1 and J respectively. L denotes the length of the stick, and L_1 the distance between the gravity center of the stick and the point O. Let $x(t)$ be the position of the cart in an inertial frame, and $\theta(t)$ the angle (taken clockwise)

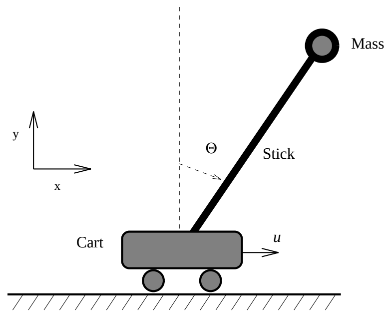

Figure 1: Scheme of the inverted pendulum

between the stick and the vertical axis.

The equations of motion are given by the Lagrange formulation in the following form

$$\left.\begin{array}{l} m_{eq}\,\ddot{x} + N\,\ddot{\theta}\,\cos\theta = u - F_1\dot{x} + N\,\dot{\theta}^2\,\sin\theta, \\ J_{eq}\,\ddot{\theta} + N\,\ddot{x}\,\cos\theta = -c\,\dot{\theta} + g\,N\,\sin\theta. \end{array}\right\}(\Sigma)$$

where $J_{eq} = J + J_1 + M_1L_1 + ML$, $m_{eq} = M + M_0 + M_1$, $N = (M_1L_1 + ML)$, c and F_1 are the friction coefficients for the motion of the cart and the pendulum. If we note the state vector $\mathbf{x}^T = (\mathrm{x}_1, x_2, x_3, x_4) = (x, \theta, \dot{x}, \dot{\theta})$, then the state representation of the inverted pendulum model is expressed (after solving the last system (Σ) with respect to $\ddot{x}$ and $\ddot{\theta}$) as follows

$$\left.\begin{array}{l} \dot{x}_1 = x_3, \\ \dot{x}_2 = x_4, \\ \dot{x}_3 = \frac{J_{eq}(-F_1x_3 + Nx_4^2\sin x_2)}{m_{eq}J_{eq} - (N\cos x_2)^2} \\ -\frac{N\cos x_2(-cx_4 + gN\sin x_2)}{m_{eq}J_{eq} - (N\cos x_2)^2} + \frac{J_{eq}}{m_{eq}J_{eq} - (N\cos x_2)^2}\,u, \\ \dot{x}_4 = \frac{-N\cos x_2(-F_1x_3 + Nx_4^2\sin x_2)}{m_{eq}J_{eq} - (N\cos x_2)^2} \\ +\frac{m_{eq}(-cx_4 + gN\sin x_2)}{m_{eq}J_{eq} - (N\cos x_2)^2} - \frac{N\cos x_2}{m_{eq}J_{eq} - (N\cos x_2)^2}\,u, \\ y_1 = x_1, \\ y_2 = x_2. \end{array}\right\}(\bar{\Sigma})$$

3. CONTROLLER DESIGN

In this section, we are interested in the design of a controller which drives the pendulum from its natural pendent position ($x_2 = \pi$) to the upright position ($x_2 = 0$) with the cart being in the desired position x_{10} at the end of the control. Recall that on one hand, the system we are treating has nonminimum phase dynamics and is partially decouplable, and on the other hand is governed by a limited source of energy. In order to overcome this difficulty and fulfill such type of stabilization, our strategy of control is as follows

At first, let us linearize the dynamics of the cart by the controller

$$u(\mathbf{x}, v) = \frac{(m_{eq}\,J_{eq} - N^2\cos{x_2}^2)\,v}{J_{eq}} + F_1\,x_3$$
$$-N{x_4}^2\sin x_2 + \frac{N\cos x_2\,(-c\,x_4 + gN\sin x_2)}{J_{eq}}, \quad (1)$$

where v is the new controller where our study will be focused on. The equations of the system (Σ) become

$$\left\{\begin{array}{l} \ddot{x} = v, \\ \ddot{\theta} = \frac{g\,N}{J_{eq}}\,\sin\theta - \frac{c}{J_{eq}}\,\dot{\theta} - \frac{N}{J_{eq}}\,\cos\theta\,v. \end{array}\right. \quad (2)$$

We can now analyze the problem of swinging-up the pendulum and the stabilization of the whole system at the desired states. The set of equations $(\bar{\Sigma})$ becomes

$$\dot{\mathbf{x}} = \begin{pmatrix} x_3 \\ x_4 \\ 0 \\ \frac{gN\sin x_2 - cx_4}{J_{eq}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{-N\cos x_2}{J_{eq}} \end{pmatrix} v \quad (3)$$

3.1. Swinging up the pendulum

The basic idea for swinging up the pendulum to the upper plane with a limited source of energy is to find a controller wich could realize gradual increase in the amplitudes of the pendulum angle x_2 after several swings. It is clear that if we dispose of a sufficient amount of energy the pendulum could be rised to its unstable equilibrium with one swing. This problem was seen either as a problem of regulation of the total energy of the pendulum [1], [19], [3] or as minimum time optimal control which leads to a bang-bang control [14], [1]. In this paper we suggest another approach which consists in finding a controller v giving back the nonlinear dynamics of the pendulum angle x_2 close to the dynamics given by the differential equation

$$\ddot{\theta} - 2\zeta\omega_n\dot{\theta} - {\omega_n}^2\sin\,\theta = 0;\; 0 < \theta < 2\pi. \quad (4)$$

This means that by fixing such dynamics, we want to increase gradually the amplitudes of x_2 until intersecting the neighborhood $\mathcal{N}_\theta = \{(x, \theta, \dot{x}, \dot{\theta}) / \left|\theta\right| \leq c_\theta\}$ where another controller is designed to stabilize the pendulum around its unstable equilibrium $x_2 = 0$, and the cart at the equilibrium $x = x_{10}$. A controller of the following type

$$v_1 = \left\{\begin{array}{ll} 0 & \text{if } c_\theta < \left|\theta\right| < \frac{\pi}{2}, \\ -a(1 - e^{-\left|\dot{\theta}\right|})\times & \\ \quad\times\mathrm{sign}(\dot{\theta}\cos\theta) & \text{if } \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}. \end{array}\right. \quad (5)$$

fulfills this aim. The philosophy behind this controller is to pump energy into the pendulum by driving it in the direction of the angular displacement θ, but only in the lower plane where $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. In the upper plane we would rather set the controller v_1 to zero so that we could intersect the neighborhood $\mathcal{N}_\theta$ with weak velocities ($\dot{\theta} \simeq 0, \dot{x} \simeq 0$) and thus while switching on the second controller v_2, only a minimum energy will be required to drive the system to its final equilibrium. Note that by applying (5) the dynamics of θ is a nonlinear second order equation given by

$$\begin{cases} \ddot{\theta} = \frac{gN}{J_{eq}} \sin\theta - \frac{c}{J_{eq}}\dot{\theta}, \\ \qquad \text{if } c_\theta < |\theta| < \frac{\pi}{2}, \\ \ddot{\theta} = \frac{gN}{J_{eq}} \sin\theta - \frac{c}{J_{eq}}\dot{\theta} + a\frac{N\cos\theta \,\text{sign}(\cos\theta)}{J_{eq}} \\ \times(1 - e^{-|\dot{\theta}|})\text{sign}(\dot{\theta}) \\ \qquad \text{if } \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}. \end{cases} \tag{6}$$

These two equations could be analyzed either by the phase plane method or by the theory of periodic orbits [10]. Notice that the term

$$-a(1 - e^{-|\dot{\theta}|})\text{sign}\,(\dot{\theta})$$

is smooth, bounded and changes its sign while $\dot{\theta}$ changes its sign, because $(1-e^{-|\dot{\theta}|})\text{sign}\,(\dot{\theta}\cos\theta)$ is an odd function of $\dot{\theta}$. We underline also that the term

$$a\prime = \frac{a\,N\cos\theta\text{sign}(\cos\theta)}{J_{eq}}$$

is bounded and positive that gives back the value of the new controller v extremely dependent on the term $(1 - e^{-|\dot{\theta}|})\text{sign}(\dot{\theta})$. Remark that for

$$\left|\dot{\theta}\right| \simeq 0, (1 - e^{-|\dot{\theta}|})\text{sign}\,(\dot{\theta}) \simeq |\dot{\theta}|\text{sign}(\dot{\theta}) = \dot{\theta},$$

(6) can be approximated by

$$\begin{cases} \ddot{\theta} = \frac{gN}{J_{eq}} \sin\theta - \frac{c}{J_{eq}}\dot{\theta} & \text{if } c_\theta < |\theta| < \frac{\pi}{2}, \\ \ddot{\theta} = \frac{gN}{J_{eq}} \sin\theta + (a\prime - c)\dot{\theta} & \text{if } \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}. \end{cases} \tag{7}$$

It is necessary to choose $a >> c$ to get a divergent response of θ, all the same when $|\dot{\theta}| \to \infty$ the quantity

$$(1 - e^{-|\dot{\theta}|})\text{sign}\,(\dot{\theta}) \simeq \text{sign}(\dot{\theta}).$$

Therefore, equations (6) could be approximated to

$$\begin{cases} \ddot{\theta} = \frac{gN}{J_{eq}} \sin\theta - \frac{c}{J_{eq}}\dot{\theta} & \text{if } c_\theta < |\theta| < \frac{\pi}{2}, \\ \ddot{\theta} = \frac{gN}{J_{eq}} \sin\theta + a\,\text{sign}(\dot{\theta}) - c\dot{\theta} & \text{if } \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}. \end{cases} \tag{8}$$

From this analysis we conclude that whenever the pendulum rotates in the lower plane its total energy is an increasing function of time. However, this latter begins to diminish under the effect of the friction force in terms of $-c\dot{\theta}^2$ when the pendulum begins to rotate in the upper plane. Let us now describe the behavior of the cart under the effect of the controller v_1. Suppose that $\Psi_\theta(t)$ is the solution of (6). This solution is a periodic orbit whose amplitudes increase with increased values of the parameter "a". The acceleration of the cart $\ddot{x} = -a\,(1 - e^{-|\dot{\Psi}_\theta(t)|})\text{sign}\,(\dot{\Psi}_\theta(t)\cos(\Psi_\theta(t)))$ is not only a bounded function of time but a periodic function as well. It results that the trajectory $x(t)$ given by the solution of the linear system $\ddot{x} = v_1$ is a also a bounded trajectory and converges to a periodic solution. The higher the value of "a", the greater of the displacement of the cart.

3.2. Stabilizing the cart and the pendulum in the upper plane

In this section we will construct a nonlinear controller wich stabilizes both the cart and pendulum around the equilibrium $x^0 = (x_{10}, 0, 0, 0)$ in the domain $\mathbf{x} \in \mathbb{R} \times [\frac{-\pi}{2} + \theta_0\,, \frac{\pi}{2} - \theta_0\,] \times \mathbb{R}^2$, where $\theta_0 = \frac{\pi}{2} - c_\theta$ is a design angle which is chosen with respect to the available amount of energy and the restricted maximum value of the cart displacement. We summarize the strategy of the controller in the following statement :

Proposition 3.1 *Consider the system $(\bar{\Sigma})$ under the feedback*

$$u(\mathbf{x}, v_2) = \frac{J_{eq}}{N\cos\,x_2}\left(\frac{g\,N}{J_{eq}}\sin\,x_2 - \frac{c}{J_{eq}}\,x_4 + \alpha\,\tan\,x_2\right.$$

$$\left. + \,\beta\,\tanh\,x_4 + e^{-\gamma\,{x_2}^2}(k(x_1 - x_{10}) + r\,x_3)\right) \tag{9}$$

in the domain $\Omega = \left\{\mathbb{R} \times \,[\frac{-\pi}{2} + \theta_0\,, \frac{\pi}{2} - \theta_0\,] \times \mathbb{R}^2\right\}$. *If the following conditions are satisfied*

- *γ is large enough, α and β are positive constants.*
- *The polynomial*

$$\lambda^4 + (\beta - r)\,\lambda^3 + \frac{\alpha\,J_{eq} - k\,J_{eq} - r\,c}{J_{eq}}\lambda^2 + \frac{r\,g\,N - k\,c}{J_{eq}}\lambda + \frac{k\,g\,N}{J_{eq}} \tag{10}$$

  is Hurwitz,

then the equilibrium point $(x_{10}, 0, 0, 0)$ is asymptotically stable. □

Proof. Consider the system $((\bar{\Sigma})$ under the feedback $u(\mathbf{x}, v_2)$. If we linearize the closed loop system around the equilibrium point $(x_{10}, 0, 0, 0)$, the dynamics of the compensated system around this equilibrim is

$$\delta\dot{\mathbf{x}} \;\;=\;\; \tilde{A}\,\delta\mathbf{x},$$

where

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k & \frac{g\,N + \alpha\,J_{eq}}{N} & r & \frac{-c + \beta\,J_{eq}}{N} \\ \frac{-N,k}{J_{eq}} & g\,\frac{N}{J_{eq}} - \frac{g\,N + \alpha\,J_{eq}}{J_{eq}} & \frac{N,r}{J_{eq}} & \frac{-c}{J_{eq}} - \frac{c + \beta\,J_{eq}}{J_{eq}} \end{pmatrix}.$$

If the coefficients $\alpha,\ \beta,\ k,\ r$ are selected so that $\tilde{A}$ is an Hurwitz matrix whose characteristic polynomial is the polynomial given in (10), then the equilibrium point $(x_{10}, 0, 0, 0)$ is asymptotically stable. We have to prove now that the unstable equilibrium of the pendulum $(\theta = 0, \dot{\theta} = 0)$ is an attractive point if $(\theta,\ \dot{\theta}) \in [\frac{-\pi}{2} + \theta_0\ , \frac{\pi}{2} - \theta_0\ ] \times \mathrm{I\!R}$. If x_2 is far from the unstable equilibrium, v_2 is approximately equal to

$$\frac{J_{eq}}{N \cos\, x_2}\left(\frac{g\, N}{J_{eq}} \sin\, x_2 + \alpha\, \tan\, x_2 + \beta\, \tanh\, x_4 - \frac{c}{J_{eq}}\, x_4\right)$$

so one can take the following candidate Lyapunov function

$$V(x_2, x_4) = -\alpha \ln\, |\cos\, x_2| + \frac{1}{2} {x_4}^2$$

and show that the time derivative of V,

$$\begin{aligned} \dot{V} &= \alpha\, x_4\, \tan\, x_2 + x_4 \left(\frac{gN}{J_{eq}} \sin\, x_2 - \frac{c}{J_e q} x_4\right) \\ &- \quad x_4 \left(\frac{g\, N}{J_{eq}} \sin\, x_2 + \beta\, \tanh\, x_4 + \alpha\, \tan\, x_2 \right. \\ &- \quad \left. \frac{c}{J_{eq}}\, x_4\right), \\ &= \quad -\beta\, x_4\, \tanh\, x_4 \le 0, \end{aligned}$$

and thus the point $(\theta,\ \dot{\theta}) = (0,\ 0)$ is an attractive point. $\diamond$

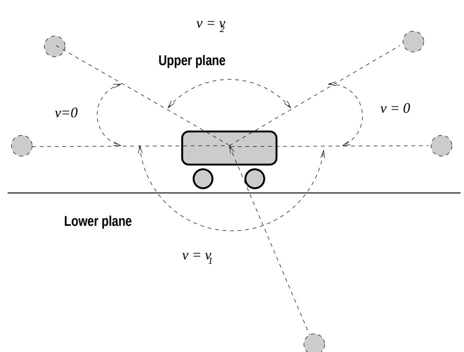

Figure 2: The strategy of the controller in the different regions.

We summarize the control strategy as $u = u(\mathbf{x}, v)$ where

$$v = \begin{cases} v_1 & \text{if } \mathbf{x} \notin \mathcal{N}_\theta, \\ v_2 & \text{if } \mathbf{x} \in \mathcal{N}_\theta. \end{cases} \tag{11}$$

4. STATE OBSERVATION

In this section we show how to estimate the unknown velocities x_3 and x_4 by a numerical procedure. Let W a moving window of the discrete observation $(y_k\ , \cdots,\ y_{k+n})$. The observation vector W is supposed to be computed with a certain error ϵ which satisfies the following conditions

$$\begin{aligned} E\left[\epsilon_i\right] &= 0, \\ E\left[\epsilon_i \epsilon_j\right] = 0, \quad i &\neq j, \\ E\left[{\epsilon_i}^2\right] &= \sigma^2. \end{aligned}$$

σ^2 denotes the variance of the random error. The numerical procedure aims at reconstructing the smooth continuous y along with its higher derivatives by considering the following minimization problem

$$\arg\min_{\hat{y}} \left\{\frac{1}{n} ||y - \hat{y}||^2 + \lambda ||T\ \hat{y}||^2\right\} \tag{12}$$

where the matrix T is an $(n - m) \times (n)$ matrix of general row

$$(-1)^{m+j-1}\ {C^{j-1}}_m \qquad j = 1, \cdots, m+1. \tag{13}$$

The norm $||.||$ denotes the Euclidian norm, and λ is a smoothing parameter chosen in the interval $[0,\ \infty[$. We look for the solution of (12) in the space of the B-spline function of order $2m$, thus the minimization problem turns out to be the following problem

$$\arg\min_{\alpha} \left\{\frac{1}{n}\, (y - B\,\alpha)^t (y - B\,\alpha) + \lambda\, \alpha^t\ B^t\ R\, B\, \alpha\right\} \tag{14}$$

such that

$$\begin{aligned} R &:= T^t\, T, \\ B_{i,j} &:= b_{j,2m}(t_i),\ i = 1, \cdots, n;\ j = 1, \cdots, n. \end{aligned}$$

and $\hat{y}$ is replaced by

$$\sum_{i=1}^{n} \alpha_i\, b_{i,2m}(t). \tag{15}$$

The vector $\alpha \in \mathrm{I\!R}^n$ is called the control vector of the spline, and $b_{i,2m}(t)$ denotes the i-th B-spline function. Minimizing (14) with respect to α, we get

$$\alpha = (n\lambda B^t\ R\, B + B^t\ B)^{-1} B^t\ y. \tag{16}$$

The smoothing parameter λ is supposed to be the minimizer of the generalized cross validation criterion

$$V(\lambda) = \frac{\frac{1}{n} ||n\lambda\, R\, B(n\lambda\, B^t\ R\, B + B^t\ B)^{-1} B^t \zeta||^2}{\left[\frac{1}{n}\text{Trace}(n\lambda\, R\, B(n\lambda\, B^t\ R\, B + B^t\ B)^{-1} B^t)\right]^2} \tag{17}$$

We refer the reader to [9] for more details about the algorithm.

5. NUMERICAL SIMULATIONS

Here, the system $(\bar{\Sigma})$ was simulated for the initial condition ${x_0}^T = (1, \pi, 0, 0)^T$ with the results depicted in the figures 3, 4, and 5. All the responses of the system are shown for the control law incorporating the observer estimates. We supposed that all the measurable states are corrupted by a zeromean noise of variance $\sigma^2 = 0.09$. The performances and the boundedness of the resulting responses illustrate the good estimates of the derivatives given by the numerical procedure discussed in Section 4. We have summarized both the system parameters and the controller parameters in the following tables

System parameters	Values
m_{eq}	1.00 Kg
J_{eq}	0.140 $\frac{N.m.s}{\text{rad}}$
N	0.340 $Kg.m$
g	10.0 m/s^2
F_1	0.01
c	0.01

Controller parameters	Values
α	57.653
β	16.456
k	3.335
r	4.456
γ	50.00
a	3.00
x_{10}	0.50 m

6. CONCLUSION

In this paper we gave a design methodology of two controllers that guarantee the stabilization of an inverted pendulum. The control strategy was based first upon linearizing the dynamics of the cart by non-interactive controller. Then by exploiting the simplicity of the resulting system, the expression of the new controller is derived to swing up the pendulum to the upper plane. In order to achieve the stabilization objective, a second nonlinear controller is proposed to steer the system to its final equilibrium. Moreover, the paper addressed the problem of observation of the velocities by a numerical method of regualarization by the B-spline functions. This approach of observation could be generalized to any highly nonlinear mechanical system where the velocities appear as unobservable states.

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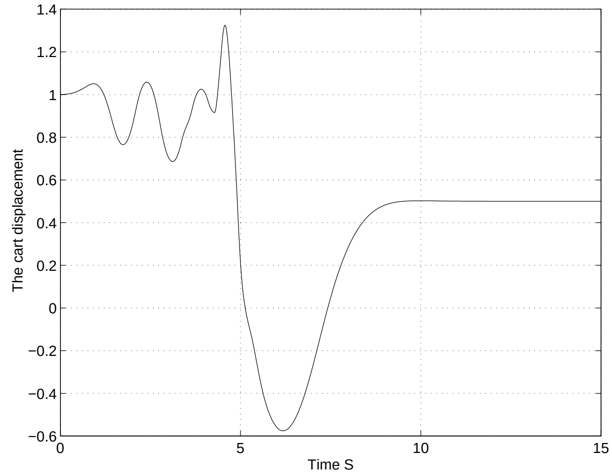

Figure 3: The cart displacement x_1

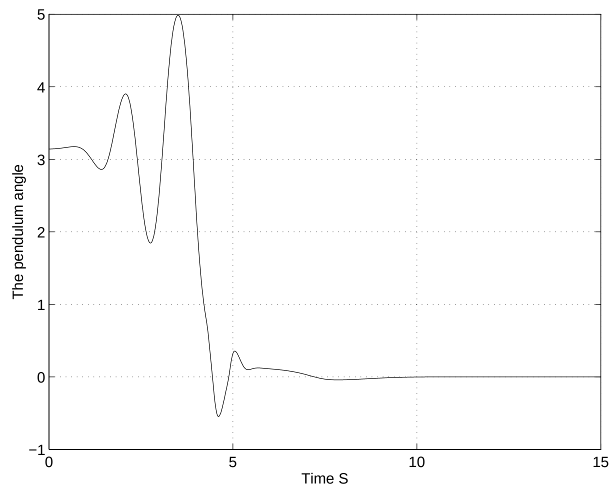

Figure 4: The pendulum angle x_2

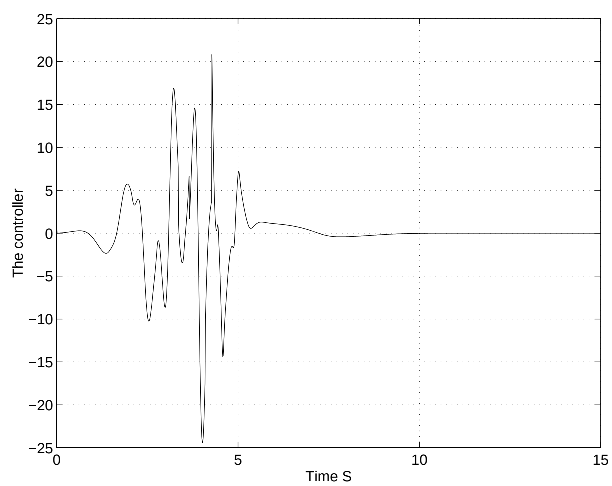

Figure 5: The controller u

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