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A NUMERICAL ALGORITHM FOR FILTERING AND STATE OBSERVATION[†]

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This paper deals with a numerical method for data fitting and estimation of continuous higher-order derivatives of a given signal from its non-exact sampled data. The proposed algorithm is a generalization of the algorithm proposed by Reinsch (1967). This algorithm is conceived as a key element in the structure of the numerical observer discussed in our recent papers. Satisfactory results are obtained which prove the efficiency of the proposed approach.

Keywords: spline functions, numerical differentiation, observers, smooth filters

1. Introduction

The problem of filtering and estimation of higher derivatives of measurable signals in the presence of noise becomes one of the principal ways to achieve control objectives, construct nonlinear observers and fulfil other physical requirements (Diop *et al.*, 1993; Diop and Ibrir, 1997; Heiss 1994; Ibrir, 1999; Ibrir and Diop, 1999; Khalil and Esfandiari, 1992; Teel and Praly, 1994; 1995). This problem has not been fully exploited yet in control and observation theory and thus it necessitates some refinements.

The numerical differentiation problem for non-exact data has received widespread attention in the literature of numerical analysis and statistics. Many algorithms were based on the regularization methods to solve ill-posed problems of numerically differentiating a signal from its discrete, potentially uncertain samples (Anderson and Bloomfield, 1974; De Boor, 1978; Craven and Wahba, 1979a; 1979b; Kimeldorf and Wahba, 1970; Reinsch, 1967; 1971; Rice and Rosenblatt 1983; Wahba, 1975; 1981; Wahba and Wold, 1975). Other approaches like kernel estimators have been considered by many researchers to estimate robust derivatives from noisy measurements. We refer the reader to the monograph (Eubank, 1988) for a survey on nonparametric regression and smoothing, and especially to works (Gasser *et al.*, 1985; Georgiev, 1984; Härdle, 1984; 1985; Müller, 1984).

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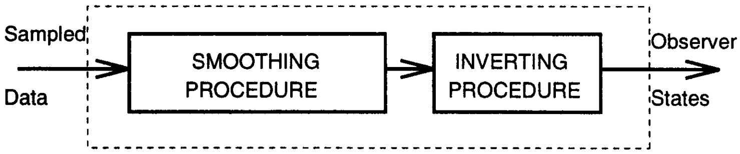

Fig. 1. A simplified scheme of the scalar numerical observer.

Recall that a numerical observer aims at reconstructing the system states from the measurement data using numerical differentiation techniques. Preliminary discussions of this type of observation were presented in (Diop *et al.*, 1993; Diop and Ibrir, 1997). The structure of the numerical observer possesses two main blocks: The first block contains a procedure which aims at smoothing noisy data with *a-priori* information about statistical properties of the noise. In the other one, an inverting procedure is implemented that takes the model equations of the system as a basis to express the remaining states in terms of the input, output, and their derivatives.

The main subject of this paper is to conceive a general smoothing algorithm to be implemented in the structure of a numerical observer. Detailed steps of the computational method will be given to evaluate continuous approximations to higher-order derivatives of a signal given by its noisy discrete values together with the filtered continuous signal. This work is related to the previous work on smoothing data by cubic spline functions developed by Reinsch (1967). In comparison with Reinsch's algorithm this paper offers a fast solution to the optimization problem with a simple discrete criterion. The solution turns out to be a spline function of an arbitrary order, fixed *a priori* by the user. Higher derivatives are then approximated by differentiating the obtained spline function.

The presented algorithm seems to be flexible because of the introduction of equivalent smoothness conditions derived from finite-difference methods. Moreover, the minimum of the functional to be considered is unique and fast convergence of Newton methods is expected. We divided our work as follows: Section 2 is devoted to the formulation of the minimization problem. In Section 3, a detailed solution to the problem is studied. Section 4 presents the approximation error analysis. Finally, the paper concludes with simulation results and further remarks.

Notation:

- $\mathbb{R}$ stands for the set of real numbers.
- $\mathbb{R}^n$ is the real vector space of real n-vectors.
- $\mathbb{R}^{n\times n}$ is the real space of real $n \times n$-matrices.
- M_n is the set of $n \times n$ complex matrices.
- If v is a vector, $||v||$ denotes the Euclidean norm of v.
- If A is a matrix, $||A|| = \max_{||x||=1} ||Ax||$.

- If A is a matrix, $\|A\|_2$ is the spectral matrix norm on M_n.
- If A is a matrix, $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{i,j}|$.
- If A is a matrix, $\rho(A) = \max\{|v_p| : v_p \text{ is an eigenvalue of } A\}$ is the spectral radius of $A \in M_n$.
- If $x(t)$ is a continuous function of time, $\|x(t)\|_\infty = \max_{t\in[a,b]} |x(t)|$.
- If $x(t)$ is a continuous function of time, $\|x(t)\|_{L_2}$ is the L_2 norm of $x(t)$.

2. Problem Formulation

Let (Σ) be a dynamical system with output ζ, and let $\begin{pmatrix}\zeta_1 & \cdots & \zeta_n\end{pmatrix}^t$ be the noisy discrete values which correspond to evenly spaced instants $(t_1, \ldots, t_n)$. One of the famous methods to smooth the nonexact data is to consider the problem

$$\text{minimize} \int_{t_1}^{t_n} \left[\widehat{\zeta}^{(m)}\right]^2 \mathrm{d}t, \tag{1}$$

subject to the constraint

$$\sum_{i=1}^{n} \left[\frac{\widehat{\zeta}(t_i) - \zeta(t_i)}{\delta\zeta_i}\right]^2 \le S, \quad \widehat{\zeta} \in C^{(m)}[t_1,\, t_n]. \tag{2}$$

Here $\widehat{\zeta}^{(m)}$ denotes the m-th derivative of the function $\widehat{\zeta}$, $\delta\zeta_i$, $i = 1, \ldots, n$ are positive numbers taken as estimates of the standard deviation in ζ_i and the number S is used to rescale $\delta\zeta_i$'s. Reinsch (1967) suggests that S could be chosen in the interval $[n - (2n)^{\frac{1}{2}},\, n + (2n)^{\frac{1}{2}}]$. We replace the last constraint by

$$\sum_{i=1}^{n} \left[\widehat{\zeta}(t_i) - \zeta(t_i)\right]^2 \le n\,\sigma^2$$

if the noise is supposed to be zero mean with variance σ^2.

Since the vector ζ is available as discrete data, in this paper we replace the continuous integral (1) by the following smoothness condition:

$$\text{minimize} \sum_{i=m}^{n-1} \left[\widehat{\zeta}_i^{(m)}\,(\Delta t)^m\right]^2, \tag{3}$$

where $\widehat{\zeta}_i^{(m)}$ stands for the finite difference scheme of the m-th derivative of $\widehat{\zeta}$ at i, and Δt means the regular forward difference of t, equal to $t_{i+1} - t_i$. Finally, the problem is formulated as follows:

$$\text{minimize} \sum_{i=m}^{n-1} \left[\widehat{\zeta}_i^{(m)}\,(\Delta t)^m\right]^2$$

subject to (2).

3. Solving the Optimization Problem by Spline Functions of Arbitrary Order

In order to compute the m-th derivative of $\widehat{\zeta}$ at point i we will only use points $\widehat{\zeta}_{i-m+1}, \widehat{\zeta}_{i-m+2}, \ldots, \widehat{\zeta}_i, \widehat{\zeta}_{i+1}$. For example, for $m = 2, 3, 4$, and 5 the smoothness conditions are

$$\sum_{i=2}^{n-1} \left[\widehat{\zeta}_{i-1} - 2\widehat{\zeta}_i + \widehat{\zeta}_{i+1}\right]^2,$$

$$\sum_{i=3}^{n-1} \left[-\widehat{\zeta}_{i-2} + 3\widehat{\zeta}_{i-1} - 3\widehat{\zeta}_i + \widehat{\zeta}_{i+1}\right]^2,$$

$$\sum_{i=4}^{n-1} \left[\widehat{\zeta}_{i-3} - 4\widehat{\zeta}_{i-2} + 6\widehat{\zeta}_{i-1} - 4\widehat{\zeta}_i + \widehat{\zeta}_{i+1}\right]^2,$$

$$\sum_{i=5}^{n-1} \left[-\widehat{\zeta}_{i-4} + 5\widehat{\zeta}_{i-3} - 10\widehat{\zeta}_{i-2} + 10\widehat{\zeta}_{i-1} - 5\widehat{\zeta}_i + \widehat{\zeta}_{i+1}\right]^2,$$

respectively. These smoothness conditions are expressed in matrix form as follows:

$$\|T\,\widehat{\zeta}\|^2 \tag{4}$$

where $\|\cdot\|$ denotes the Euclidean norm. For $m = 2, 3, 4$ we have

$$T_{(n-2)\times n} = \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix},$$

$$T_{(n-3)\times n} = \begin{bmatrix} -1 & 3 & -3 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & -3 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 3 & -3 & 1 \end{bmatrix},$$

$$T_{(n-4)\times n} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 \end{bmatrix}.$$

Luckily, we can generalize the form of T as an $(n-m)\times n$ matrix of the general row

$$(-1)^{m+j-1}\frac{m!}{(j-1)!(m-j+1)!}, \quad j = 1, \ldots, m+1, \tag{5}$$

and the solution to (1) and (2) turns out to be the minimum of the functional

$$J := \widehat{\zeta}^t T^t T \widehat{\zeta} + \lambda \left\{ (\zeta - \widehat{\zeta})^t D^{-2} (\zeta - \widehat{\zeta}) + \mu^2 - S \right\}, \tag{6}$$

where λ is the Lagrange multiplier and μ is an auxiliary variable, $D^{-2} = \mathrm{diag}(\delta\zeta_1^{-2}, \dots, \delta\zeta_n^{-2})$. Looking for the minimum of (6) in the space of the B-spline functions of order $k = 2m$, we replace $\widehat{\zeta}$ by

$$\sum_{i=1}^{n} \alpha_i b_{i,2m}(t), \tag{7}$$

such that $\alpha = (\alpha_i,\ i = 1, \dots, n) \in \mathbb{R}^n$, and $b_{i,2m}$ is the i-th positive B-spline function.

We write J in terms of the control vector α as follows:

$$J := \alpha^t B^t T^t T B\alpha + \lambda \left\{ (\zeta - B\,\alpha)^t D^{-2} (\zeta - B\,\alpha) + \mu^2 - S \right\},$$

with

$$B_{n\times n} = \begin{bmatrix} b_{1,k}(t_1) & b_{2,k}(t_1) & \cdots & b_{n,k}(t_1) \\ b_{1,k}(t_2) & b_{2,k}(t_2) & \cdots & b_{n,k}(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_{1,k}(t_n) & b_{2,k}(t_n) & \dots & b_{n,k}(t_n) \end{bmatrix}.$$

The minimum of the functional $J(\alpha, \mu, \lambda)$ is obtained by differention with respect to α, μ and λ, and then equating the result with zero. This gives

$$(T^t T + \lambda D^{-2})\, B\,\alpha - \lambda\, D^{-2} \zeta = 0, \tag{8}$$

$$2\mu\lambda = 0, \tag{9}$$

$$(\zeta - B\,\alpha)^t D^{-2} (\zeta - B\,\alpha) + \mu^2 - S = 0. \tag{10}$$

Let u be an $(n-m) \times 1$ vector such that

$$D^2\, T^t\, u = \zeta - B\,\alpha. \tag{11}$$

Substituting (11) in (8), we get

$$(T^t T + \lambda D^{-2})(\zeta - D^2\, T^t\, u) = \lambda\, D^{-2} \zeta \tag{12}$$

and hence

$$(T\, D^2\, T^t + \lambda\, I) u = T\, \zeta, \tag{13}$$

where I is the $(n-m) \times (n-m)$ identity matrix. From (13) it follows that

$$u(\lambda) = (T\, D^2\, T^t + \lambda\, I)^{-1}\, T\, \zeta, \tag{14}$$

and the control vector α is

$$\alpha(\lambda) = B^{-1} \left(\zeta - D^2\, T^t\, u(\lambda) \right). \tag{15}$$

The Lagrange multiplier λ must not be equal to zero. From (9) we conclude that

$$\mu = 0 \tag{16}$$

and

$$\left(\zeta - B\,\alpha(\lambda)\right)^t D^{-2}\left(\zeta - B\,\alpha(\lambda)\right) = S. \tag{17}$$

Clearly, λ has to satisfy (17). Then the control point of the spline will be obtained using (14) and (11). Note that

$$F^2(\lambda) := (\zeta - B\alpha)^t D^{-2}(\zeta - B\alpha) = \|D^{-1}(\zeta - B\alpha)\|^2 = \|DT^t u\|^2.$$

If we set $Q = D\,T^t$, then λ is obtained as the solution to the nonlinear equation

$$u^t(\lambda)\,Q^t\,Q\,u(\lambda) = S. \tag{18}$$

By the application of the Newton method, the root λ_r of (17) is obtained after a limited number of the following iterations:

$$\lambda_{k+1} = \lambda_k - 2\frac{F^2(\lambda_k)}{\dfrac{\mathrm{d}F^2(\lambda_k)}{\mathrm{d}\lambda}}\left[\frac{\sqrt{F^2(\lambda_k)}}{\sqrt{S}} - 1\right].$$

We have

$$\frac{\mathrm{d}F^2}{\mathrm{d}\lambda} = 2u^t\,Q^t\,Q\,\frac{\mathrm{d}u}{\mathrm{d}\lambda} = -\,2u^t\,Q^t\,Q\,(T\,D^2\,T^t + \lambda\,I)^{-1}u$$

while $F^2(\lambda_k) > S$, so the Newton iteration is

$$\lambda_{k+1} = \lambda_k + \frac{u^t\,Q^t\,Q\,u}{u^t\,Q^t\,Q\,(T\,D^2\,T^t + \lambda_k\,I)^{-1}u}\left[\frac{\sqrt{u^t\,Q^t\,Q\,u}}{\sqrt{S}} - 1\right] \tag{19}$$

Remark 1. The matrix $(T\,D^2\,T^t + \lambda\,I)$ is invertible for any $\lambda \geq 0$.

Theorem 1. *The spline function (7) as the solution to the constrained optimization problem (1) and (2) is unique.*

Proof. The proof of this theorem is based on the fact that $F^2(\lambda)$ is strictly decreasing in λ because the matrix $-Q^t\,Q\,(T\,D^2\,T^t + \lambda\,I)^{-1}$ is negative definite for all $\lambda \geq 0$. Consequentely, the root of the nonlinear equation $F^2(\lambda) = S$ is unique. ∎

Newton's iteration involves at each step the calculation of the inverse of the matrix $(T\,D^2\,T^t + \lambda\,I)$. In order to accelerate the rate of convergence, we compute the inverse of the matrix $(T\,D^2\,T^t + \lambda\,I)$ with the use of the Leverrier algorithm. We have

$$(T\,D^2\,T^t + \lambda\,I)^{-1} = \frac{\sum\limits_{i=1}^{n-m} R_{i-1}\,\lambda^{n-m-i}}{\sum\limits_{i=0}^{n-m} \rho_i\,\lambda^{n-m-i}} \tag{20}$$

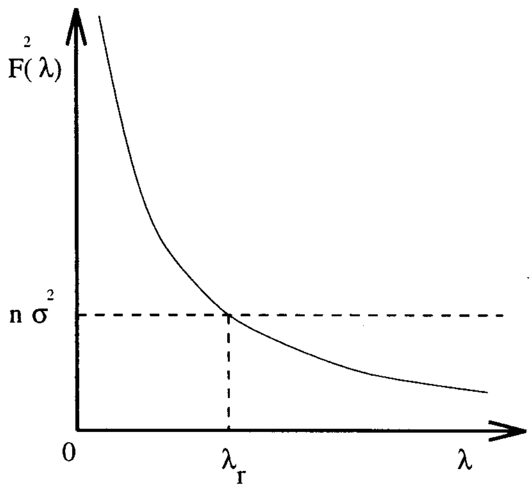

Fig. 2. Function $F^2(\lambda)$ and the root λ_r of $F^2(\lambda) = n\sigma^2$.

such that

$$\rho_i := \frac{1}{i}\,\mathrm{trace}\left[T\,D^2\,T^t\,R_{i-1}\right], \tag{21}$$

$$R_i := \rho_i I - T\,D^2\,T^t\,R_{i-1}. \tag{22}$$

where I is the $(n-m)\times(n-m)$ identity matrix. The matrices R_i $(i = 0,\dots,n-m-1)$ and the coefficients ρ_i $(i = 0,\dots,n-m)$ should be computed before starting the Newton iteration. Accordingly, fast convergence is expected.

4. Error Bounds

At first, we present the method of computing the optimal knot sequence if the number of observations n and the degree of smoothness m are given. The number of knots is fixed at $n + 2m$ and the computational method of calculating the optimal knot sequence is inspired by the scheme of Micchelli, Rivlin and Winograd (De Boor, 1978). If we write $\tau = (\tau_1,\dots,\tau_{n+2m})$, then

$$\tau_1 = \tau_2 = \cdots = \tau_{2m} = t_1,$$

$$\tau_{n+1} = \tau_{n+2} = \cdots = \tau_{n+2m} = t_n,$$

The $n-2m$ interior knots in $[t_1, t_n]$ are choosen to be the breakpoints of the unique step function h for which

- $|h(t)| = 1$ for all $t \in [t_1, t_n]$ with $\lim_{\delta t\to 0} h(t_1 + \delta t) = 1$,
- $h(t)$ has $\leq n-2m$ sign changes in $[t_1, t_n]$, and

- $\int_{t_1}^{t_n} f(t)h(t)\,\mathrm{d}t = 0$ for every $f \in \{S_{2m} = \mathrm{span}\,(B_{i,2m})$, linear space of splines of order $2m\}$.

The $n-2m$ interior knots for given observations $\{t_1, \ldots, t_n\}$ are computed using Newton's method with the initial guess:

$$t_{2m+i} = (t_{i+1} + \cdots + t_{i+2m-1})\,/(2m-1), \quad i = 1, \ldots, n-2m.$$

For a further analysis we assume that the regularization parameter is obtained by a numerical procedure of root finding. According to the amount of noise which corrupts the measurements, we distinguish three cases.

Case 1. (A low noise level) In the sequel, we take $D = I$ and $S = n\sigma^2$. In this case the amount of noise is small, i.e. the optimal value of λ is large. Since the error between the noisy samples and values of the smoothing spline at the breakpoints is given by

$$\zeta - B\alpha = T^t(TT^t + \lambda I)^{-1}T\zeta, \tag{23}$$

we deduce that

$$\|\zeta - B\alpha\|_\infty \le \|T\|_\infty^2 \left\|(TT^t + \lambda I)^{-1}\right\|_\infty \|\zeta\|_\infty. \tag{24}$$

Since

$$(TT^t + \lambda I)^{-1} = \frac{1}{\lambda}\left(\frac{1}{\lambda}TT^t + I\right)^{-1}$$

and the spectral radius of the matrix $\left(\frac{1}{\lambda}TT^t\right)$ is supposed to be less than 1, we can expand $\left(\frac{1}{\lambda}TT^t + I\right)^{-1}$ in power series:

$$\left(\frac{1}{\lambda}TT^t + I\right)^{-1} = \sum_{\ell=0}^{\infty}(-1)^\ell\left(\frac{1}{\lambda}TT^t\right)^\ell, \tag{25}$$

Then

$$\left\|\left(\frac{1}{\lambda}TT^t + I\right)^{-1}\right\|_\infty \le \sum_{\ell=0}^{\infty}\left\|\left(\frac{1}{\lambda}TT^t\right)^\ell\right\|_\infty \le \sum_{\ell=0}^{\infty}\left\|\left(\frac{1}{\lambda}TT^t\right)\right\|_\infty^\ell = \frac{1}{1 - \left\|\left(\frac{1}{\lambda}TT^t\right)\right\|_\infty}$$

Finally, we have the error bound

$$\|\zeta - B\alpha\|_\infty \le \frac{\frac{1}{\lambda}\|T\|_\infty^2}{1 - \left\|\left(\frac{1}{\lambda}TT^t\right)\right\|_\infty}\|\zeta\|_\infty = \frac{\frac{1}{\lambda}\left[\sum_{j=1}^{m+1}\left|C_m^{j-1}\right|\right]^2}{1 - \frac{1}{\lambda}\sum_{j=1}^{m+1}\sum_{i=1}^{m+1}\left|C_m^{j-1}C_m^{i-1}\right|}\|\zeta\|_\infty.$$

Note that

- $\left\|TT^t\right\|_\infty$ increases with increasing values of the observation n for a fixed smoothing parameter m.
- $\left\|(TT^t+\lambda I)^{-1}\right\|_\infty$ decreases with increasing values of λ for fixed values of n and m.

Case 2. (A high noise level) In this case $\lambda \to 0$. We have

$$(TT^t+\lambda I)^{-1} = \left(I+\lambda(TT^t)^{-1}\right)^{-1} (TT^t)^{-1}. \tag{26}$$

The spectral radius $\rho\left(\lambda(TT^t)^{-1}\right) < 1$. Consequently, we develop the matrix $\left(I+\lambda(TT^t)^{-1}\right)^{-1}$ in power series as follows:

$$\left(I+\lambda(TT^t)^{-1}\right)^{-1} = \sum_{\ell=0}^{\infty}(-1)^\ell\left[\lambda(TT^t)^{-1}\right]^\ell. \tag{27}$$

Hence

$$\|\zeta - B\alpha\|_\infty \le \frac{\|T\|_\infty^2\left\|(TT^t)^{-1}\right\|_\infty}{1-\left\|\lambda(TT^t)^{-1}\right\|_\infty}\|\zeta\|_\infty. \tag{28}$$

Case 3. In case the smoothing parameter is neither too small nor too large, we can always find an error bound by computing the matrix S composed of the eigenvectors of TT^t and put the expression of the error in the following form:

$$\zeta - B\alpha = T^t S(\Lambda+\lambda I)^{-1}S^{-1}T\,\zeta, \tag{29}$$

such that

$$S^{-1}TT^tS = \Lambda = \operatorname{diag}\left(\widehat{\lambda}_1 \quad \cdots \quad \widehat{\lambda}_{n-m}\right).$$

Finally, we obtain

$$\|\zeta - B\alpha\|_\infty \le \frac{\|T\|_\infty^2\,\|S\|_\infty\,\left\|S^{-1}\right\|_\infty}{\min\limits_{i=1,\,n-m}\left(\widehat{\lambda}_i+\lambda\right)}\|\zeta\|_\infty \tag{30}$$

If we denote by $s(t)$ the resulting continuous spline function of degree $2m-1$ with control vector α, then the errors made while approximating a higher-order derivative, by differentiating $s(t)$, are bounded:

$$\left\|\tilde{\zeta}^{(\ell)}(t) - s^{(\ell)}(t)\right\|_\infty \le \frac{1}{\ell!}\frac{m!}{\sqrt{m}}\,\delta^{m-\ell-\frac{1}{2}}\left\|\tilde{\zeta}^{(m)}\right\|_{L_2[t_1,\,t_n]}, \tag{31}$$

such that $\tilde{\zeta}^{(\ell)}$ is the exact ℓ-th derivative of the function $\tilde{\zeta}(t)$ which best fits the exact measurements without noise, and $\delta = \max\{\Delta\tau_i,\ i = 1, \ldots, n+2m\}$. From these bounds we conclude that the interpolation error decreases if the values of the smoothing parameter m and the number of the measurements n increase indefinitely. The number of knots is determined according to these two parameters. The errors made on the interpolating spline and its derivatives are also sensitive to the mesh size of the optimal knot sequence, cf. (31).

5. Algorithm

Summarizing, the ultimate computational procedure is as follows:

- For a selected order $2m$ and given breakpoints $(t_1, \ldots, t_n)$ construct an optimal knot sequence $(\tau)_1^n$, and the corresponding B-spline bases $b_{i,2m}, i = 1, \ldots, n$.

- From the matrix B such that

$$B_{i,j} := b_{j,2m}(t_i), \quad i = 1, \ldots, n, \quad j = 1, \ldots, n.$$

- Compute the matrix T such that

$$T_{i,j} := \begin{cases} (-1)^{m+j-i} C_m^{j-i} & \text{for } i = 1, \ldots, n-m \text{ and } j = i \ldots, m+i, \\ 0 & \text{otherwise.} \end{cases}$$

- Compute the matrix

$$D^{-2} := \operatorname{diag}(\delta\zeta_1^{-2}, \ldots, \delta\zeta_n^{-2}).$$

- Compute the matrices $Q := DT^t$ and R_i, $i = 0, \ldots, n-m-1$ with the coefficients ρ_i, $i = 0, \ldots, n-m$ using (20)–(22). If the noise is zero mean with variance σ^2, replace the matrix D by the identity matrix and S by $n\sigma^2$.
- Compute the root of the nonlinear equation $F^2(\lambda) = S$ using (18)–(20).
- Compute the vector u from (14).
- Solve the linear system

$$B\,\alpha = (\zeta - D^2\,T^t\,u)$$

  with respect to the control points of the spline α. Since the matrix B is positive definite, we write $B = \bar{R}^t\,\bar{R}$ for the Cholesky factorization of B. We have to solve

$$\bar{R}^t\,y = (\zeta - D^2\,T^t\,u)$$

with respect to y, and then

$$\bar{R}\,\alpha = y$$

with respect to α.

- Compute the derivatives of the spline using the formulae

$$D^j\Big(\sum_i \alpha_i b_{i,k}\Big) = \sum_i \alpha_i^{j+1} b_{i,k-j}$$

with

$$\alpha_r^{j+1} := \begin{cases} \alpha_r & \text{for } j = 0, \\ \dfrac{1}{k-j}\,\dfrac{\alpha_r^j - \alpha_{r-1}^j}{t_{r+k-j} - t_r} & \text{for } j > 0. \end{cases} \tag{32}$$

6. Example

Here, we consider the system

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2, \\ \dot{\zeta}_2 &= -150\big(1+\cos(t)\big)\,\zeta_1 - 10\big(2+\sin(t)\big)\,\zeta_2, \\ y &= \zeta_1 + w, \end{aligned}$$

where the scalar output y is supposed to be corrupted by zero-mean white noise with variance $\sigma^2 = 0.0012$. We consider that the measurements are collected with an even step $\Delta t = 0.01S$.

Remark 2. Notice that even though the measurements are available at discrete instants, the numerical observer does not require a sampled-data representation of the system.

In the simulations presented below, the order of the spline is $k = 2m = 6$ and the number of noisy points is $n = 151$. Figure 3 shows the filtered continuous output ζ with discrete noisy output. In Figs. 4 and 5 we respectively show the first and the second derivative of the filtered solution along the exact derivatives. (By the exact derivatives we mean, the solution of the system considered without additive noise.)

Using our algorithm, we realize that the Newton method converges after 21 iterations, and the Lagrange parameter is approximately 0.0207.

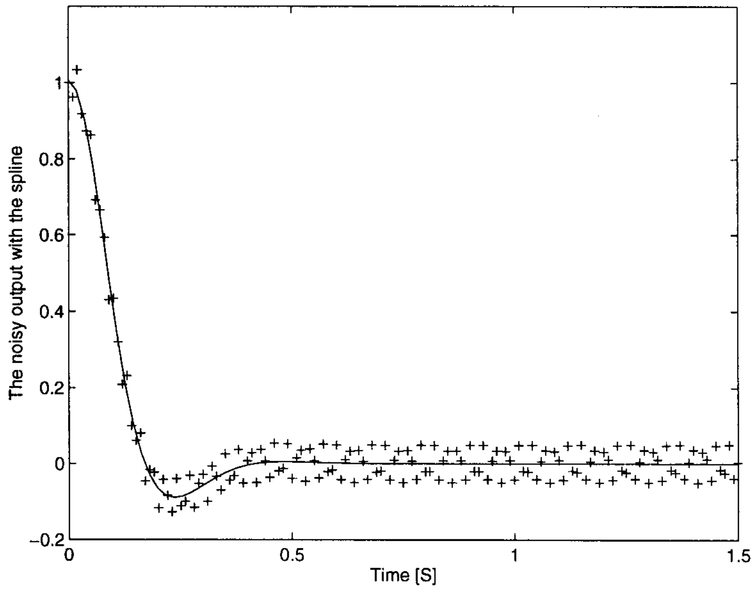

Fig. 3. The filtered output (solid line) and the noisy output (+).

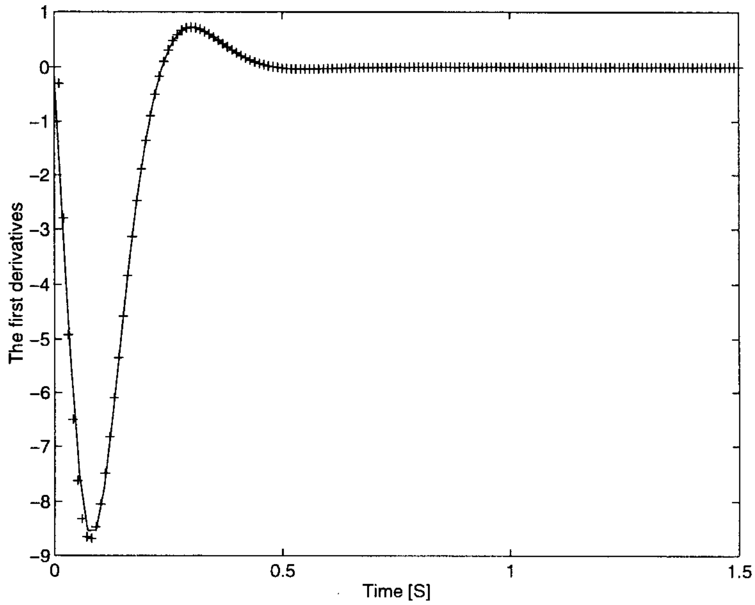

Fig. 4. The exact first derivative (+) and the derivative of the spline (solid).

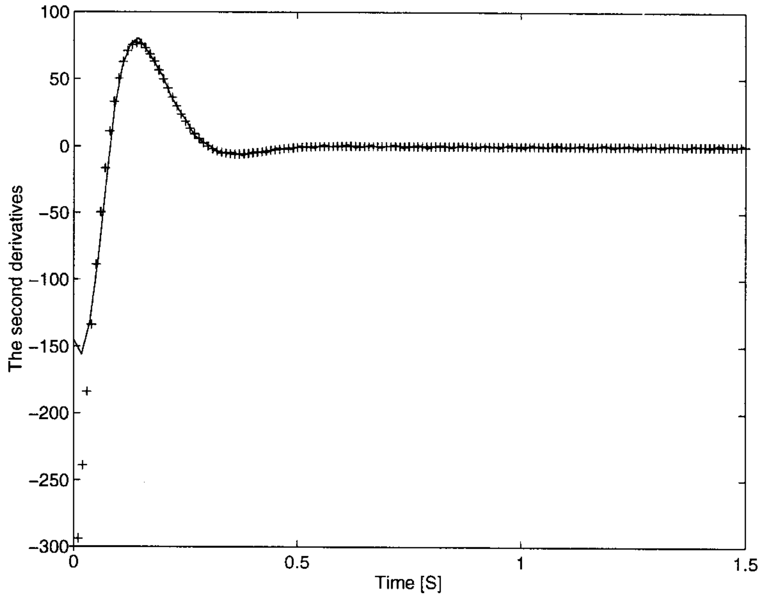

Fig. 5. The exact second derivative (+) and the derivative of the spline (solid).

7. Conclusions

Based on *a-priori* knowledge of the nature of the noise, the steps of a numerical algorithm used as a filter and an observer were examined. The design problem has been formulated in such a manner that finding the coefficients of the smooth function and its derivatives requires solving a simple constrained optimization problem. The simplicity of the criterion to be minimized comes from the fact that new conditions of smoothness are proposed. In order to solve the design problem, solution to a nonlinear equation and to some linear systems is required. In comparison with the algorithms studied in the cited literature, our procedure guarantees a unique solution to the optimization problem with a simple discrete criterion. Moreover, it reduces the computation burden and thereby yields a potentially valuable tool to design on-line state estimators. Finally, it is possible to extend the idea of the equivalent conditions of smoothness to solve the classical regularization problem discussed in Craven and Wahba, 1979b. It is also possible to choose the regularization parameter in such a manner that it is independent of the statistical properties of noise and only depends on the measurement.

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