# Convex Optimization Approach to Observer-Based Stabilization of Uncertain Linear Systems


**Salim Ibrir**
Department of Mechanical & Industrial Engineering,
Concordia University,
1455 de Maisonneuve Blvd. West,
Montreal, Quebec H3G 1M8, Canada
e-mail: sibrir@encs.concordia.ca



*New sufficient linear matrix inequality conditions guaranteeing the stability of uncertain linear systems by means of dynamic output feedbacks are presented. It is shown that the search of an observer-based controller for this class of systems is fundamentally decomposed into two main problems: robust stability with a memoryless state feedback and observer design with measured uncertainties. Under the fulfilment of the developed linear matrix inequalities conditions, we show that the observer-based problem is solvable without any need for some equality constraints or iterative computational algorithms. Examples showing the potential of the results are presented.* [DOI: 10.1115/1.2363202]



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## 1 Introduction

Generally, controller design strategies are based on the knowledge of all state variables of the considered system. Since the measurement of the whole state vector is in most cases not available for feedback, the use of observers is of great importance and still an unavoidable task to solve the desired control issue. Actually, state reconstruction is not limited to control exercises, but is also quite important to detect faults, monitor performance, or identify some unknown parameters of dynamical systems. We mean by observer a dynamical system whose states converge asymptotically to the real ones when time elapses. If the process model is in the form of a system of linear differential equations, then the problem of constructing an observer is essentially solved by the Kalman filter [1] and the Luenberger observer [2]. For a system of uncertain differential equations, however, there is no generic solution, which is the reason for extensive research in this area for the past decades (see, for example, [3,4]). When parts of the system dynamic are not completely known and the state vector is not entirely available for feedback, the available results are limited to some cases including matched uncertainties [5], norm-bounded uncertainties [6,7], and uncertainties of dyadic types [8]. Even with the existence of considerable efforts to cope with the observer-based issue of uncertain systems, the design of converging observers for uncertain systems remains an open and a challenging problem (see, for example, [9]).

When the observer is used in closed-loop configurations, the proof of the system stability is not a trivial task. Roughly speaking, the determination of the observer and the controller gains is usually conditioned by the solution of a nonconvex optimization problem. Consequently, the available solutions to this problem are generally stated as iterative linear matrix inequality conditions or as constrained convex optimization conditions that involve some equality constraints. Equality constraint as used in [10] permits us to reverse the observer-based issue to a convex one, but, in the meantime, it may increase the conservatism of the conditions under the presence of significant uncertainties. In this paper we propose new sufficient linear matrix inequality conditions that guarantee the stability of uncertain linear systems under the action of dynamic output feedbacks. The proposed conditions are not subject to any equality constraints and are numerically solvable by any commercially LMI software. It will be shown that the existence of stabilizing observer-based feedback is related to the solution of two linear matrix inequality conditions. The first sufficient LMI condition describes the existence of the observer, and the second one stands for the possibility of stabilizing the system with a memoryless state feedback. A numerical example is introduced to show the efficiency of the developed results.

Throughout this paper, the notation $A>0$ (respectively $A<0$) means that the matrix $A$ is positive definite (respectively negative definite). We denote by $A'$ the matrix transpose of $A$. We note by $\boldsymbol{I}$ and $\mathbf{0}$ the identity matrix and the null matrix of appropriate dimensions, respectively. $\mathbb{R}$ stands for the sets of real numbers, and "$\star$" is used to notify a matrix element that is induced by transposition.

## 2 Preliminary Results

*Fact* 1. For given matrices $X$ and $Y$ with appropriate dimensions, we have

$$X'Y + Y'X \leqslant \beta X'X + \frac{1}{\beta}Y'Y, \quad \beta>0 \tag{1}$$

Henceforth, the result of the Schur complement lemma is used to prove the main result of this paper. Therefore, we would rather recall it [11].

LEMMA 1. *Given constant matrices $M$, $N$, $Q$ of appropriate dimensions where $M$ and $Q$ are symmetric, then $Q>0$ and $M+N'Q^{-1}N<0$ if and only if*

$$\begin{bmatrix} M & N' \\ N & -Q \end{bmatrix} < 0$$

*or equivalently*

$$\begin{bmatrix} -Q & N \\ N' & M \end{bmatrix} < 0$$

In order to use the result of Lemma 4, it is necessary to recall the following lemmas.

LEMMA 2. *Let $S$, $Y$, and $Z$ be given $k\times k$ symmetric matrices such that $S\geqslant 0$, $Y<0$, and $Z\geqslant 0$. Furthermore, assume that*

$$(\eta' Y\eta)^2 - 4(\eta' S\eta)(\eta' Z\eta) > 0 \tag{2}$$

*for all nonzero $\eta\in\mathbb{R}^k$. Then there exists a constant $\lambda$ such that the matrix $\lambda^2 S+\lambda Y+Z$ is negative definite.*

*Proof*. For the proof, see [12].

LEMMA 3. *Given any $x\in\mathbb{R}^n$ and four matrices $P$, $D$, $F$, $G$ of appropriate dimensions, we have*

$$\max\{(x'PDFGx)^2 : F'F \leqslant \boldsymbol{I}\} = x'PDD'Pxx'G'Gx \tag{3}$$

*Proof*. See Ref. [12] for the proof.

Throughout this paper, the results of the following lemmas are extensively used in the proof of the main result of this paper.

LEMMA 4. *Let $Y$ be a symmetric matrix and let $H$, $E$, and $F$ be three arbitrary matrices with appropriate dimensions. Then the following linear matrix inequality holds*


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$$Y + HFE + E'F'H' < 0 \tag{4}$$

*for all F satisfying* $F'F \leq \boldsymbol{I}$, if and only if there exists a scalar $\varepsilon > 0$ such that

$$Y + \varepsilon HH' + \varepsilon^{-1}E'E < 0 \tag{5}$$

*Proof.* This lemma is a direct consequence of Lemmas 2 and 3.

(i) Necessity. Suppose that there exist $Y<0$ and $H$, $E$, $F$ satisfying (4) with $F'F \leq \boldsymbol{I}$. Then for any $\eta \in \mathbb{R}^k$, where $k$ is the rank of $Y$ and $\eta \neq 0$, we have

$$\eta' Y \eta < -\eta'(HFE + E'F'H')\eta \tag{6}$$

This implies that

$$\eta' Y \eta < -2\max\{\eta'(HFE)\eta : F'F \leq \boldsymbol{I}\} < 0 \tag{7}$$

Hence,

$$(\eta' Y \eta)^2 > 4\max\{(\eta'(HFE)\eta)^2 : F'F \leq \boldsymbol{I}\} \tag{8}$$

By Lemma 3, we get

$$(\eta' Y \eta)^2 > 4(\eta' HH' \eta)(\eta' E'E \eta) \tag{9}$$

Using the result of Lemma 4, we can say that there exists $\varepsilon > 0$ such that

$$\varepsilon^2 HH' + \varepsilon Y + E'E < 0 \tag{10}$$

which is exactly inequality (5).

(ii) Sufficiency. Suppose now that there exists $\epsilon > 0$ such that inequality (5) holds. By the use of fact 1, we can write that there exists $\varepsilon$ such that

$$Y + HFE + E'F'H' \leq Y + \varepsilon HH' + \varepsilon^{-1}E'E \tag{11}$$

By our above supposition $Y + \varepsilon HH' + \varepsilon^{-1}E'E < 0$, we conclude that for any $F$, such that $F'F \leq \boldsymbol{I}$, we have

$$Y + HFE + E'F'H' < 0 \tag{12}$$

This ends the proof.

LEMMA 5. *For any* $\epsilon > 0$ *and symmetric and positive definite matrix P such that* $P - \epsilon\boldsymbol{I}$ *is a full rank matrix, we have*

$$\epsilon^2 P^{-1} > -P + 2\epsilon\boldsymbol{I} \tag{13}$$

*Proof.* Since $P - \epsilon\boldsymbol{I}$ is a full rank matrix, and $P > 0$, then,

$$(P - \epsilon\boldsymbol{I})' P^{-1} (P - \epsilon\boldsymbol{I}) > 0 \tag{14}$$

By factorization of the last inequality, we obtain (13).

## 3 Observer-Based Control of Uncertain Linear Systems

Consider the uncertain linear system

$$\begin{aligned} \dot{\xi}(t) &= (E + \Delta E)\xi(t) + (F + \Delta F)u(t) \\ y(t) &= (G + \Delta G)\xi(t) + (H + \Delta H)u(t) \end{aligned} \tag{15}$$

where $\xi(t):[0\ \infty) \mapsto \mathbb{R}^n$ is the state vector, $u(t):[0\ \infty) \mapsto \mathbb{R}^m$ is the control input, and $y(t):[0\ \infty) \mapsto \mathbb{R}^p$ is the system output. The nominal matrices $E \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times m}$, $G \in \mathbb{R}^{p\times n}$, and $H \in \mathbb{R}^{p\times m}$ are constant known matrices and $\Delta E \in \mathbb{R}^{n\times n}$, $\Delta F \in \mathbb{R}^{n\times m}$, $\Delta G \in \mathbb{R}^{p\times n}$, and $\Delta H \in \mathbb{R}^{p\times m}$ are partially known uncertainties defined as follows

$$\begin{aligned} \Delta E &= M_E F_E(\xi,t) N_E; \quad F'_E(\xi,t) F_E(\xi,t) \leq \boldsymbol{I} \\ \Delta F &= M_F F_F(\xi,t) N_F; \quad F'_F(\xi,t) F_F(\xi,t) \leq \boldsymbol{I} \\ \Delta G &= M_G F_E(\xi,t) N_G; \quad F'_G(\xi,t) F_G(\xi,t) \leq \boldsymbol{I} \\ \Delta H &= M_H F_E(\xi,t) N_H; \quad F'_H(\xi,t) F_H(\xi,t) \leq \boldsymbol{I} \end{aligned} \tag{16}$$

where $M_E \in \mathbb{R}^{n\times n}$, $N_E \in \mathbb{R}^{n\times n}$, $M_F \in \mathbb{R}^{n\times m}$, $N_F \in \mathbb{R}^{m\times m}$, $M_G \in \mathbb{R}^{p\times n}$, $N_G \in \mathbb{R}^{n\times n}$, $M_H \in \mathbb{R}^{p\times m}$, and $N_H \in \mathbb{R}^{m\times m}$ are known matrices and $F_E(\xi,t)$, $F_F(\xi,t)$, $F_G(\xi,t)$, and $F_H(\xi,t)$ are some unknown matrices of appropriate dimensions. We assume that the pairs $(E,F)$ and $(E,G)$ are controllable and observable, respectively. Let us consider the new state variables

$$\begin{aligned} &x(t) = \begin{bmatrix} \xi(t) \\ u(t) \end{bmatrix}, \quad A = \begin{bmatrix} E & F \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad B = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}, \quad C = [G\ H] \\ &\Delta A = \begin{bmatrix} \Delta E & \Delta F \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad \Delta C = [\Delta G\ \Delta H] \end{aligned} \tag{17}$$

and define $v(t) = \dot{u}(t)$ as the new control input. Then, we have

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + Bv(t) \\ y(t) &= (C + \Delta C)x(t) \end{aligned} \tag{18}$$

where $\Delta A = N_A F_A(x,t) N_A$, $\Delta C = M_C F_C(x,t) N_C$, and

$$\begin{aligned} &M_A = \begin{bmatrix} M_E & M_F \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad N_A = \begin{bmatrix} N_E & \boldsymbol{0} \\ \boldsymbol{0} & N_F \end{bmatrix} \\ &M_C = [M_G\ M_H], \quad N_C = \begin{bmatrix} N_G & \boldsymbol{0} \\ \boldsymbol{0} & N_H \end{bmatrix} \end{aligned} \tag{19}$$

$$F_A(x,t) = \begin{bmatrix} F_E(x,t) & \boldsymbol{0} \\ \boldsymbol{0} & F_F(x,t) \end{bmatrix}, \quad F_C(x,t) = \begin{bmatrix} F_G(x,t) & \boldsymbol{0} \\ \boldsymbol{0} & F_H(x,t) \end{bmatrix}$$

This technique, which consists of adding $m$ integrators to system (15), permits us to regroup uncertainties in the state matrix and makes the output totally free from the control input. The disappearance of uncertainties from the input matrix is utterly compensated by both the appearance of this uncertainty in the state matrix and the augmentation of the system order. Consequently, the structure of uncertainties is being modified. Besides the robustness that we gain by the input integral action, it is worthwhile to mention that this uncertainty regroupment technique is essentially used to facilitate the decomposition of the observer-based issue into two separate problems.

The main objective is to find a stabilizing controller $v(t) = Y_1 P_1^{-1}\hat{x}(t)$ such that system (18) is globally asymptotically stable, where $\hat{x}(t)$ is the state vector of the following observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + BY_1P_1^{-1}\hat{x}(t) + P_2^{-1}Y_2[C\hat{x}(t) - y(t)] \tag{20}$$

The matrices $P_1 \in \mathbb{R}^{(n+m)\times(n+m)}$, $P_2 \in \mathbb{R}^{(n+m)\times(n+m)}$ are symmetric and positive definite matrices; $Y_1 \in \mathbb{R}^{m\times(n+m)}$ and $Y_2 \in \mathbb{R}^{(n+m)\times p}$ are arbitrary real matrices to be determined. The design of both the feedback and the observer gains is summarized in the following statement.

THEOREM 1. *Consider system* (15) *and observer* (20). *Then if there exist two symmetric and positive definite matrices* $P_1 \in \mathbb{R}^{(n+m)\times(n+m)}$, $P_2 \in \mathbb{R}^{(n+m)\times(n+m)}$, *two real matrices* $Y_1 \in \mathbb{R}^{m\times(n+m)}$, $Y_2 \in \mathbb{R}^{(n+m)\times p}$, *and five strictly positive constants* $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, $\alpha$, *and* $\beta$ *such that the following linear matrix inequalities hold for* $0<\varepsilon_1<1$, $0<\varepsilon_2<1$

$$\begin{bmatrix} -P_1 & \boldsymbol{I} \\ \boldsymbol{I} & -(2\beta-\alpha)\boldsymbol{I} \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} A'P_2 + P_2A + Y_2C + C'Y_2' & \beta\boldsymbol{I} & P_2M_A & Y_2M_C \\ \star & -P_1 & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & -\varepsilon_1\boldsymbol{I} & \boldsymbol{0} \\ \star & \star & \star & -\varepsilon_2\boldsymbol{I} \end{bmatrix} < 0 \tag{22}$$

$$\begin{bmatrix} P_1A' + AP_1 + Y_1'B' + BY_1 + \varepsilon_3 M_A M_A' & -BY_1 & P_1N_A' & P_1N_C' & P_1N_A' \\ \star & -\alpha\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & -(2-\varepsilon_1)\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & \star & -(2-\varepsilon_2)\boldsymbol{I} & \boldsymbol{0} \\ \star & \star & \star & \star & -\varepsilon_3\boldsymbol{I} \end{bmatrix} < 0 \tag{23}$$

*then for any initial conditions* $\hat{x}(0)\neq 0$, *the observer-based controller* $u(t)=\int_0^t Y_1P_1^{-1}\hat{x}(\tau)\,\mathrm{d}\tau$ *is a stabilizing feedback for system* (15).

*Proof*. Let $e(t)=x(t)-\hat{x}(t)$. Then, we have

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A + BY_1P_1^{-1} + \Delta A & -BY_1P_1^{-1} \\ \Delta A + P_2^{-1}Y_2\Delta C & A + P_2^{-1}Y_2C \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{24}$$

Define

$$V(x(t),e(t)) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}' \begin{bmatrix} P_1^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & P_2 \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{25}$$

as a Lyapunov function candidate to the dynamics (24). Then, we obtain

$$\dot{V}(x(t),e(t)) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}' \begin{bmatrix} \mathcal{M}_{1,1} & \mathcal{M}_{1,2} \\ \mathcal{M}'_{1,2} & \mathcal{M}_{2,2} \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{26}$$

where

$$\mathcal{M}_{1,1} = (A+\Delta A)'P_1^{-1} + P_1^{-1}(A+\Delta A) + P_1^{-1}(BY_1 + Y_1'B')P_1^{-1}$$

$$\mathcal{M}_{1,2} = \Delta A'P_2 + \Delta C'Y_2' - P_1^{-1}BY_1P_1^{-1} \tag{27}$$

$$\mathcal{M}_{2,2} = A'P_2 + P_2A + Y_2C + C'Y_2'$$

Evidently, $\dot{V}(x(t),e(t))<0$ if

$$\begin{bmatrix} \mathcal{M}_{1,1} & \mathcal{M}_{1,2} \\ \mathcal{M}'_{1,2} & \mathcal{M}_{2,2} \end{bmatrix} < 0 \tag{28}$$

Pre- and postmultiplying the last matrix inequality by

$$\begin{bmatrix} P_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{29}$$

we obtain

$$\begin{bmatrix} P_1\mathcal{M}_{1,1}P_1 & P_1\mathcal{M}_{1,2} \\ \mathcal{M}'_{1,2}P_1 & \mathcal{M}_{2,2} \end{bmatrix} < 0 \tag{30}$$

The last matrix inequality can be rewritten as follows

$$\begin{bmatrix} P_1\mathcal{M}_{1,1}P_1 & -BY_1P_1^{-1} \\ -P_1^{-1}Y_1'B' & \mathcal{M}_{2,2} \end{bmatrix} + \begin{bmatrix} P_1N_A' \\ \boldsymbol{0} \end{bmatrix} F_A'(x,t)[\boldsymbol{0}\ M_A'P_2] + \begin{bmatrix} \boldsymbol{0} \\ P_2M_A \end{bmatrix} F_A(x,t)[N_AP_1\ \boldsymbol{0}] + \begin{bmatrix} P_1N_C' \\ \boldsymbol{0} \end{bmatrix} F_C'(x,t)[\boldsymbol{0}\ M_C'Y_2'] + \begin{bmatrix} \boldsymbol{0} \\ Y_2M_C \end{bmatrix} F_C(x,t)[N_CP_1\ \boldsymbol{0}] < 0 \tag{31}$$

Using the result of Lemma 4, we conclude that the last matrix inequality holds if there exist two positive constants $\varepsilon_1$ and $\varepsilon_2$ such that the following holds

$$\begin{bmatrix} P_1\mathcal{M}_{1,1}P_1 + \varepsilon_1 P_1N_A'N_AP_1 + \varepsilon_2 P_1N_C'N_CP_1 & -BY_1P_1^{-1} \\ \star & \mathcal{M}_{2,2} + \varepsilon_1^{-1}P_2M_AM_A'P_2 + \varepsilon_2^{-1}Y_2M_CM_C'Y_2' \end{bmatrix} < 0 \tag{32}$$

For any $\alpha>0$, the matrix inequality (32) is rewritten as

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & P_1^{-1} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \mathcal{G}_{1,1} & -BY_1 & \boldsymbol{0} \\ -Y_1'B' & -\alpha\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \mathcal{G}_{2,2} + \alpha P_1^{-1}P_1^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & P_1^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} < 0 \tag{33}$$

where $\mathcal{G}_{1,1}=P_1\mathcal{M}_{1,1}P_1+\varepsilon_1 P_1 N_A' N_A P_1+\varepsilon_2 P_1 N_C' N_C P_1$ and $\mathcal{G}_{2,2}=\mathcal{M}_{2,2}+\varepsilon_1^{-1}P_2 M_A M_A' P_2+\varepsilon_2^{-1}Y_2 M_C M_C' Y_2'$. From inequality (33), we can say that inequality (32) holds if the following matrix inequalities hold simultaneously

$$\begin{bmatrix} \mathcal{G}_{1,1} & -BY_1 \\ -Y_1'B' & -\alpha\boldsymbol{I} \end{bmatrix} < 0 \tag{34}$$

$$\mathcal{G}_{2,2}+\alpha P_1^{-1}P_1^{-1} < 0 \tag{35}$$

From inequality (34), we deduce that the first condition to ensure the stability of system (18), under the action of an observer-based feedback, is to verify that

$$P_1(A+\Delta A)'+(A+\Delta A)P_1+BY_1+Y_1'B' < -Q1 \tag{36}$$

where $Q_1=\alpha^{-1}BY_1Y_1'B'+\varepsilon_1 P_1 N_A' N_A P_1+\varepsilon_2 P_1 N_C' N_C P_1$. This condition implies that there exists a robust memoryless state feedback $v=Kx(t)$ that stabilizes the system $\dot{x}(t)=(A+\Delta A)x(t)+Bv(t)$. Since inequality (35) can be rewritten as

$$A'P_2+P_2A+Y_2C+C'Y_2' < -Q_2 \tag{37}$$

where $Q_2=\alpha P_1^{-1}P_1^{-1}+\varepsilon_1^{-1}P_2 M_A M_A' P_2+\varepsilon_2^{-1}Y_2 M_C M_C' Y_2'$, then we conclude that (37) is a sufficient condition for the existence of an observer for the dynamics (18) where the link to inequality (36) is quantified by the presence of the term $\alpha P_1^{-1}P_1^{-1}$ in $Q_2$. Now, in order to make these separate conditions linear with respect to their variables, let $\beta$ be some positive real constant such that

$$P_1 > \frac{\alpha}{\beta^2}\boldsymbol{I} \tag{38}$$

Then, by the Schur complement, inequality (38) is equivalent to

$$\begin{bmatrix} -P_1 & \boldsymbol{I} \\ \boldsymbol{I} & -\dfrac{\beta^2}{\alpha}\boldsymbol{I} \end{bmatrix} < 0 \tag{39}$$

Using the result of Lemma 5, we can write that $-(\beta^2/\alpha)\boldsymbol{I} \leq (-2\beta+\alpha)\boldsymbol{I}$. Consequently, a sufficient condition to fulfill (38) is

$$\begin{bmatrix} -P_1 & \boldsymbol{I} \\ \boldsymbol{I} & -(2\beta-\alpha)\boldsymbol{I} \end{bmatrix} < 0 \tag{40}$$

From inequality (38), we have

$$\alpha P_1^{-1}P_1^{-1} < \beta^2 P_1^{-1} \tag{41}$$

Then (35) is verified if $\mathcal{G}_{2,2}+\beta^2 P_1^{-1} < 0$, which is equivalent by the Schur complement to inequality (22). By the use of the result of Lemma 4, then $P_1\mathcal{M}_{1,1}P_1 < 0$ if and only if there exists $\varepsilon_3 > 0$ such that

$$P_1A'+AP_1+\varepsilon_3 M_A M_A'+\varepsilon_3^{-1}P_1 N_A' N_A P_1+Y_1'B'+BY_1 < 0 \tag{42}$$

From (42), we obtain another sufficient condition for the fulfillment of (34), that is

$$\begin{bmatrix} P_1A'+AP_1+Y_1'B'+BY_1+\varepsilon_3 M_A M_A' & -BY_1 & P_1N_A' & P_1N_C' & P_1N_A' \\ \star & -\alpha\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & -\varepsilon_1^{-1}\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & \star & -\varepsilon_2^{-1}\boldsymbol{I} & \boldsymbol{0} \\ \star & \star & \star & \star & -\varepsilon_3\boldsymbol{I} \end{bmatrix} < 0 \tag{43}$$

Since for all $0<\varepsilon_1<1$ and $0<\varepsilon_2<1$, we can write $-\varepsilon_1^{-1}\boldsymbol{I} \leq -(2-\varepsilon_1)\boldsymbol{I}$, and $-\varepsilon_2^{-1}\boldsymbol{I} \leq -(2-\varepsilon_2)\boldsymbol{I}$, then a sufficient condition to fulfill (43) is inequality (23). This ends the proof.

The passage from inequality (32) to the sufficient conditions (34) and (35) is certainly paid by a certain conservatism. However, the appearance of two independent positive constants $\alpha$ and $\beta$ relieves the degree of conservatism of the LMIs. It is important to outline that there is no restrictive assumption on the choice of these parameters and hence, more degree of freedom is available to impose other optimality constraints. The parameters $(\varepsilon_i)_{1\leq i\leq 3}$ have appeared as "if and only if conditions," except the constraints that we have imposed on the choice of $\varepsilon_1$ and $\varepsilon_2$.

*Remark* 1. The positive parameters $\alpha$ and $\beta$ are introduced in order to dissociate the observer-based control problem into two separate linear matrix inequalities problems. The parameter $\alpha$ is used essentially to divide the nonconvex observer-based control issue into two separate problems. However, the parameter $\beta$ is introduced to make the two separate sufficient conditions linear with respect to their variables.

It is always interesting to impose a certain degree of stability to satisfy some practical requirements. By putting $z(t)=e^{\gamma t}x(t)$, where $\gamma>0$, then system (18) is equivalent to the following

$$\dot{z}(t)=(A+\gamma\boldsymbol{I}+\Delta A)z(t)+e^{\gamma t}Bv(t)$$
$$y(t)=(C+\Delta C)z(t) \tag{44}$$

where the output $y(t)$ in (18) is replaced by $(C+\Delta C)z(t)$. By constructing an observer for the $z$ system and by the use of Theorem 1, we can derive new stability conditions for the $z$ system by replacing the matrix $A$ in inequalities (22) and (23) by $A+\gamma\boldsymbol{I}$. Hence, for a given $\gamma$, if there exist $P_1=P_1'>0$, $P_2=P_2'>0$, $Y_1$, $Y_2$, $0<\varepsilon_1<1$, $0<\varepsilon_2<1$, $\varepsilon_3>0$, $\alpha>0$, $\beta>0$ that verify inquality (21) along with the following linear matrix inequalities

$$\begin{bmatrix} \mathcal{K}_{1,1} & \beta\boldsymbol{I} & P_2M_A & Y_2M_C \\ \star & -P_1 & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & -\varepsilon_1\boldsymbol{I} & \boldsymbol{0} \\ \star & \star & \star & -\varepsilon_2\boldsymbol{I} \end{bmatrix} < 0 \tag{45}$$

$$\begin{bmatrix} \mathcal{L}_{1,1} & -BY_1 & P_1N_A' & P_1N_C' & P_1N_A' \\ \star & -\alpha\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & -(2-\varepsilon_1)\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & \star & -(2-\varepsilon_2)\boldsymbol{I} & \boldsymbol{0} \\ \star & \star & \star & \star & -\varepsilon_3\boldsymbol{I} \end{bmatrix} < 0 \tag{46}$$

where $\mathcal{K}_{1,1}=2\gamma P_2+A'P_2+P_2A+Y_2C+C'Y_2'$ and $\mathcal{L}_{1,1}=2\gamma P_1+P_1A'+AP_1+Y_1'B'+BY_1+\varepsilon_3 M_A M_A'$, then a stabilizing controller of the form

$$v(t) = Y_1 P_1^{-1}\hat{x}(t) = e^{-\gamma t} Y_1 P_1^{-1}\hat{z}(t) \tag{47}$$

will ensure a prescribed degree of stability of the $x$ system (18). Furthermore, when $\Delta A \equiv 0$ and $\Delta C \equiv 0$, the real parts of the eigenvalues of the closed loop system will be lower or equal to $-\gamma$. Depending upon the maximum value of the uncertainty norm, one can choose an appropriate $\gamma$ that defines the decay rate of the system and the observer dynamics.

*Remark* 2. If $\Delta F \equiv 0$, $H \equiv 0$, and $\Delta H \equiv 0$, there is no need to augment system (15) with $m$ integrators. In this case, a stabilizing controller of the form $u(t) = K\hat{\xi}(t)$ can be built by following the same steps as we have developed for the general case.

## 4 Illustrative Examples

As an example, consider the following uncertain system

$$\dot{\xi}(t) = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \Delta A\right)\xi(t) + \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \Delta B\right)u(t)$$
$$y(t) = ([1\ 1] + \Delta G)\xi(t) + (1 + \Delta H)u(t) \tag{48}$$

where

$$M_E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_E = \begin{bmatrix} 0 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad M_F = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix}, \quad N_F = 0.2$$
$$M_G = [0.1\ 0.3], \quad N_G = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0 \end{bmatrix}, \quad M_H = 0.3, \quad N_H = 0.3 \tag{49}$$

A solution to the linear matrix inequalities (21), (22), and (23) is

$$P_1 = \begin{bmatrix} 1.6590 & -0.0633 & 0.2037 \\ -0.0633 & 1.5313 & -0.9621 \\ 0.2037 & -0.9621 & 2.1599 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 3.6368 & -0.0245 & 2.4829 \\ -0.0245 & 0.6504 & -1.7887 \\ 2.4829 & -1.7887 & 8.7524 \end{bmatrix}$$

$$Y_1 = [0.8427\ \ -0.9860\ \ -2.0311], \quad Y_2 = \begin{bmatrix} -1.3942 \\ -2.7632 \\ -1.3510 \end{bmatrix} \tag{50}$$

$$\varepsilon_1 = 0.9824, \quad \varepsilon_2 = 0.6152, \quad \varepsilon_3 = 0.6133$$

$$\alpha = 2.0314, \quad \beta = 1.6274$$

In order to show that the proposed LMI are not conservative, let us introduce the following example with more significant uncertainties (the maximum rate of the system parameters change is of 50%)

$$\dot{\xi}(t) = \left(\begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} + \Delta E\right)\xi(t) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}u(t)$$
$$y(t) = \left(\begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix} + \Delta G\right)\xi(t) \tag{51}$$

where

$$M_E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.8 & 0 \\ 0 & 2.8 & 0 \end{bmatrix}, \quad N_E = \begin{bmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0 \\ 0 & 0.4 & 0.2 \end{bmatrix}$$
$$M_G = \begin{bmatrix} 0.9 & 0 & 0.45 \\ 0.45 & 0 & 0 \end{bmatrix}, \quad N_G = \begin{bmatrix} 0.25 & 0.5 & 0 \\ -0.25 & 0 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} \tag{52}$$

As reported in Remark 1, we are not in need of augmenting the system with an integrator. It is sufficient to replace $A$, $B$, $C$, $M_A$, $N_A$, $M_C$, and $N_C$ in inequalities (22) and (23) by $E$, $F$, $G$, $M_E$, $N_E$, $M_G$, and $N_G$, respectively. The solutions are

$$P_1 = \begin{bmatrix} 2.3705 & -0.8387 & 0.8074 \\ -0.8387 & 1.1650 & -1.1111 \\ 0.8074 & -1.1111 & 2.2520 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 27.9999 & -8.6113 & 8.3629 \\ -8.6113 & 10.7629 & -10.6267 \\ 8.3629 & -10.6267 & 10.7471 \end{bmatrix}$$

$$Y_1 = [-2.3958\ 3.6771\ 1.4301], \quad Y_2 = \begin{bmatrix} 1.4569 & -3.3126 \\ 8.9808 & -22.1436 \\ 2.7672 & -7.2938 \end{bmatrix} \tag{53}$$

$$\varepsilon_1 = 0.6192, \quad \varepsilon_2 = 0.9209, \quad \varepsilon_3 = 0.1075$$

$$\alpha = 6.0201, \quad \beta = 4.2575$$

## 5 Conclusion

In this note, we presented new sufficient linear matrix inequality conditions for the dynamic output feedback stabilization of uncertain linear systems. In contrast with existing results in this subject, the proposed conditions are novel, in the sense that they are not attached to any equality constraint or iterative steps and hence, they could be implemented with ease in any commercial LMI software. In addition, the presented method deals with general uncertain systems that exhibit uncertainties in all the nominal matrices. The extension of this result to stabilize uncertain systems with more optimality conditions is under investigation. This point shall be the subject of our next contribution.


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