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Brief paper

Observer-based control of a class of time-delay nonlinear systems having triangular structure[☆]

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ABSTRACT

Time-delay systems constitute a special class of dynamical systems that are frequently present in many fields of engineering. It has been shown in the literature that the existence of a stabilizing observer-based controller is related to delay-dependent conditions that are generally satisfied for a small time delay. Motivating works towards reducing the conservatism of the results are among the on-going research topics especially when partial-state measurements are imposed. This paper investigates the problem of observer-based stabilization of a class of time-delay nonlinear systems written in triangular form. First, we show that a delay nonlinear observer is globally convergent under the global Lipschitz condition of the system nonlinearity. Then, it is shown that a parameterized linear feedback that uses the observer states can stabilize the system whatever the size of the delay. An illustrative example is provided to approve the theoretical results.

1. Introduction

Time delays have always been among the most difficult problems encountered in many fields of engineering. These delays could occur for various reasons and in different magnitudes. It may attributed to delay in measurement, delay in control or delay in one or many states of the system being controlled or observed. Therefore, the ubiquitousness of time delays in many practical systems such as tele-operation systems, communication networks, process control systems, robotic manipulators, and electrical circuits will continue to pose a strong challenge for control engineers and theoreticians.

Unlike systems governed by ordinary differential equations, delay systems are infinite dimensional in nature and time-delay is, in many cases, a source of instability, see for example Bellen, Guglielmi, and Ruehli (1999), Dugard and Verriest (1997), Gu, Vladimir, and Chen (2003), Hale and Lunel (1993), Niculescu (2001), and the references therein. Therefore, the presence of delays makes the system analysis and control design much more complicated specially when partial state measurements are available. During the last decade, significant developments in intelligent and robust control of time-delay systems have been appeared (Dugard & Verriest, 1997; Gu et al., 2003; Niculescu, 2001). Observer analysis of such kind of systems has been the subject of numerous papers and monographs, see e.g., Boutayeb (2001), Germani, Manes, and Pepe (2001), Germani, Manes, and Pepe (2002), Ibrir, Xie, and Su (2006), Marquez, Moog, and Velasco Villa (2002), Wang, Goodall, and Burnham (2002), Xu and Dooren (2002); Zhou, Xiao, and Lu (2009) and the references therein. Referring to the extensive results on stabilization and observation of time-delay systems, the conditions under which stabilizing feedbacks exist are classified into two possible categories: delay-independent conditions and delay-dependent ones (Mahmoud, 2000). Even, delay-independent conditions do not take into account the size of the delay and may lead to robust control design, the delay-dependent results reveal less conservative than delay-independent conditions. Nevertheless, in most cases, a small delay is tolerable to maintain stability by output feedbacks.

In this paper a nonlinear-observer-based controller is designed to steer the states of a class of nonlinear time-delay systems to the origin, irrespective of the size of the delay. By exploiting the triangular structure of the system and, by choosing an appropriate Lyapunov–Krasovskii functional, we show that a high-gain parameterized linear controller achieves the global asymptotic stability. It is shown that the parameterized gains of the feedback depend on the Lipschitz constant of the system nonlinearity and depend only on one parameter. An illustrative example is studied to show the efficiency and the performance of the proposed observer-based controller.

In Section 2 the system description along with some preliminary results are presented. The theory of the observer-based controller will be the topic of Section 3 where a detailed proof of

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the system stability is presented. In Section 4 simulation results are shown to highlight the usefulness of the theoretical results. Throughout this paper, we note by $\mathbb{R}$ the set of real numbers. The notation $A > 0$ (resp. $A < 0$), with A being a matrix, means that the matrix A is positive definite (resp. negative definite). A' is the matrix transpose of A. "$\star$" is used to notify an element which is induced by transposition. $\triangleq$ stands for an equality by definition. $\lambda_k(A)$ stands for the k-th eigenvalue of the matrix A. $\|\cdot\|$ is the Euclidean norm, $\|\cdot\|_\infty$ is the infinity norm, and Spec(A) stands for the set of eigenvalues of the matrix A. C_n^k is the binomial coefficient. The notation $y^{(i)}(t)$ stands for the i-th derivative of $y(t)$ with respect to time. The sequence $z_1, z_2, \ldots, z_k$ is noted by $(z_i)_{1\le i\le k}$. $\delta_{i,j}$ is the Kronecker symbol and the notation x_t denotes the state $x(s)$ defined on the time interval $[t-\tau,\ t]$ where τ stands for the constant time delay. It should be noted that the arguments of state variables are omitted when no confusion may arise.

2. System description and preliminary results

Consider the nonlinear time-delay system having the following lower triangular structure:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) + f_1(x_1(t), x_1(t-\tau)),\\
\dot{x}_2(t) &= x_3(t) + f_2(x_1(t), x_2(t), x_1(t-\tau), x_2(t-\tau)),\\
&\vdots\\
\dot{x}_i(t) &= x_{i+1}(t) + f_i(x_1(t), \ldots, x_i(t), x_1(t-\tau), \ldots, x_i(t-\tau)),\\
&\vdots\\
\dot{x}_n(t) &= f_n(x_1(t), \ldots, x_n(t), x_1(t-\tau), \ldots, x_n(t-\tau)) + u(t),\\
y(t) &= x_1(t),
\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the control input, and $y(t) \in \mathbb{R}$ is the only measured output. We assume that the delay τ is constant and $x(t) = \phi(t)$ for $t \le \tau$. The main objective of the paper is to conceive a nonlinear observer-based controller that stabilizes system (1) to the origin for all initial conditions. As a matter of fact, system (1) is uniformly observable with respect to the input $u(t)$; therefore, the input cannot disturb in anyway the observability of the time-delay system. To check this property, let us begin by analyzing the first dynamical equation of the system, that is

$$\dot{x}_1(t) = x_2(t) + f_1(x_1(t), x_1(t-\tau)). \tag{2}$$

Since $y(t) = x_1(t) = \varphi_1(y(t))$ is measured; $\varphi_1(s) = s$ then, the unmeasured state $x_2(t)$ can be expressed in terms of $y(t)$, $y(t-\tau)$ and $\dot{y}(t)$ as

$$\begin{aligned}
x_2(t) &= \dot{y}(t) - f_1(y(t), y(t-\tau))\\
&= \varphi_2(y(t), y(t-\tau), \dot{y}(t)).
\end{aligned} \tag{3}$$

The next unmeasured state $x_3(t)$ is then extracted from the second dynamical equation of the system, i.e.,

$$\begin{aligned}
x_3(t) &= \dot{\varphi}_2(y(t), y(t-\tau), \dot{y}(t))\\
&\quad - f_2(y(t), \varphi_2(y(t), y(t-\tau), \dot{y}(t)), y(t-\tau),\\
&\qquad \varphi_2(y(t-\tau), y(t-2\tau), \dot{y}(t-\tau)))\\
&= \varphi_3(y(t), y(t-\tau), y(t-2\tau), \dot{y}(t), \dot{y}(t-\tau), \ddot{y}(t)).
\end{aligned} \tag{4}$$

By recursive calculation and based upon the algebraic expressions (3)–(4), the i-th unmeasured state $x_i(t)$ can be easily expressed in terms of $(y(t-k\tau))_{0\le k\le i-1}$, $(\dot{y}(t-k\tau))_{0\le k\le i-2}$, $(\ddot{y}(t-k\tau))_{0\le k\le i-3}$, ..., $y^{(i-1)}(t)$. The process of calculation continues until one arrives at the last state $x_n(t)$ that can be expressed in terms of the output, the delayed output, and their respective higher derivatives. More explicitly, there exists a functional φ_n such that

$$\begin{aligned}
x_n(t) = \varphi_n(&(y(t-k\tau))_{0\le k\le n-1}, (\dot{y}(t-k\tau))_{0\le k\le n-2},\\
&(\ddot{y}(t-k\tau))_{0\le k\le n-3}, \ldots, y^{(n-1)}(t)).
\end{aligned} \tag{5}$$

Since the first functionals $(\varphi_i)_{1\le i\le n-1}$ are input free and $x_n(t)$ is extracted from the $n-1$-dynamical equation that does not depend on the input $u(t)$ then, the input $u(t)$ has no effect on the observability of the states $(x_i(t))_{1\le i\le n}$. Notice that when the system nonlinearities are continuously differentiable then, the resulting functionals $(\varphi_i)_{2\le i\le n}$ that define the observability of the system states are also continuous everywhere. This fact directly implies the uniform observability of the unmeasured states for any input $u(t)$.

In matrix form, the time-delay system (1) takes the following form:

$$\begin{aligned}
\dot{x}(t) &= A\,x(t) + f(x(t), x(t-\tau)) + B\,u(t),\\
y(t) &= C\,x(t),
\end{aligned} \tag{6}$$

where

$$f(x(t), x(t-\tau)) \triangleq \begin{bmatrix} f_1(x_1(t), x_1(t-\tau)) \\ f_2(x_1(t), x_2(t), x_1(t-\tau), x_2(t-\tau)) \\ \vdots \\ f_n(x(t), x(t-\tau)) \end{bmatrix} \in \mathbb{R}^n,$$

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \qquad B \triangleq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n, \tag{7}$$

$$C \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}' \in \mathbb{R}^n.$$

To complete the description of the time-delay system the following assumptions are taken into consideration.

Assumption 1. The nonlinearity $f(x(t), x(t-\tau))$ is smooth, globally Lipschitz, and well-defined for all $x(t) \in \mathbb{R}^n$ with $f(0,0) = 0$.

Assumption 2. For all $t \ge 0$, the delay τ is known and constant.

Lemma 1. *Let $W(\gamma)$ and $P(\gamma)$ be the solutions of the following matrix equations*

$$\begin{aligned}
&-\gamma W(\gamma) - W(\gamma)A' - AW(\gamma) + \gamma BB' = 0,\\
&-\gamma R(\gamma) - A'R(\gamma) - R(\gamma)A + \gamma C'C = 0,
\end{aligned} \tag{8}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^n$ are defined as in (6). Then, for any $\gamma > 0$,

(i) *the matrix $W^{-1}(\gamma)$ and $R(\gamma)$ are explicitly given by:*

$$W^{-1}(\gamma) = T_w'(\gamma)\,T_w(\gamma), \qquad R(\gamma) = T_r'(\gamma)\,T_r(\gamma), \tag{9}$$

where the (i,j)-entry of $T_w(\gamma)$ is given by

$$(T_w(\gamma))_{i,j} \triangleq C_{i-1}^{n-j}\,\gamma^{n-j}, \qquad (T_r(\gamma))_{i,j} \triangleq (-1)^{i+j}\,\frac{C_{j-1}^{i-1}}{\gamma^{j-1}},$$
$$1 \le i, j \le n. \tag{10}$$

(ii) *For any $\gamma > 1$, and any real lower triangular matrix $\mathscr{L}$, there exist four positive constants c_1, c_2, c_3 and c_4 such that*

$$\|T_w(\gamma)\,\mathscr{L}\,T_w^{-1}(\gamma)\| \le c_1 + \frac{c_2}{\gamma}$$
$$\|T_r(\gamma)\,\mathscr{L}\,T_r^{-1}(\gamma)\| \le c_3 + \frac{c_4}{\gamma}. \tag{11}$$

Proof. The matrix $W(\gamma)$ is explicitly given by:

$$W(\gamma) = \int_0^\infty \gamma\, e^{-\gamma\, t} e^{-A\, t}\, BB'\, e^{-A'\, t}\mathrm{d}t. \tag{12}$$

Since the matrix A is nilpotent for $k \ge n$ then,

$$e^{A\,t} = \sum_{k=0}^{\infty} \frac{A^k}{k!}\, t^k = \sum_{k=0}^{n-1} \frac{A^k}{k!}\, t^k. \tag{13}$$

Since (A, B) is a controllable pair then, from (12), we conclude that $W(\gamma) > 0$ for all $\gamma > 0$. More explicitly,

$$W(\gamma) = \int_0^\infty \gamma\, e^{-\gamma\, t} \left[\sum_{k=0}^{n-1} \frac{A^k}{k!}\, t^k\right] BB' \left[\sum_{k=0}^{n-1} \frac{A^k}{k!}\, t^k\right]' \mathrm{d}t$$
$$= \left((-1)^{2n-i-j}\frac{C_{2n-i-j}^{n-i}}{\gamma^{2n-i-j}}\right)_{1\le i,j\le n}. \tag{14}$$

Starting from (10), the (i, j)-entry of the inverse of the matrix $T_w(\gamma)$ is given by

$$(T_w^{-1}(\gamma))_{i,j} = (-1)^{i+j-1}\frac{C_{n-i}^{j-1}}{\gamma^{n-i}}. \tag{15}$$

This implies that the (i, j)-entry of the matrix $T_w^{-1}(\gamma)T_w'^{-1}(\gamma)$ is

$$(T_w^{-1}(\gamma)T_w'^{-1}(\gamma))_{i,j} = \frac{(-1)^{i+j}}{\gamma^{2n-i-j}} \sum_{k=1}^{n} (-1)^{2(k-1)} C_{n-i}^{k-1} C_{n-j}^{k-1}$$
$$= \frac{(-1)^{i+j}}{\gamma^{2n-i-j}} \sum_{k=1}^{n} C_{n-i}^{k-1} C_{n-j}^{k-1}$$
$$= \frac{(-1)^{i+j}}{\gamma^{2n-i-j}}\, C_{2n-i-j}^{n-i}$$
$$= \frac{(-1)^{2n-i-j}}{\gamma^{2n-i-j}}\, C_{2n-i-j}^{n-i}$$
$$= (W(\gamma))_{i,j}. \tag{16}$$

According to this result, by forming the matrix product $T_w'(\gamma)T_w(\gamma) = W^{-1}(\gamma)$ an explicit formulae of $W^{-1}(\gamma)$ is given for $1 \le i, j \le n$

$$(W^{-1}(\gamma))_{i,j} = (T_w'(\gamma)T_w(\gamma))_{i,j}$$
$$= \sum_{k=1}^{n} (C_{k-1}^{n-i}\gamma^{n-i})(C_{k-1}^{n-j}\gamma^{n-j})$$
$$= \gamma^{2n-i-j} \sum_{k=1}^{n} C_{k-1}^{n-i} C_{k-1}^{n-j}. \tag{17}$$

Similarly, the matrix $R(\gamma)$ is given by

$$\int_0^\infty \gamma\, e^{-\gamma\, t} e^{-A'\, t}\, C'C\, e^{-A\, t}\mathrm{d}t. \tag{18}$$

Using the fact that the pair (A, C) is observable and $A^k = 0$ for $k \ge n$ then, $\forall \gamma > 0$

$$R(\gamma) = \int_0^\infty \gamma\, e^{-\gamma\, t} \left[\sum_{k=0}^{n-1} \frac{A^k}{k!}\right]' C'C \left[\sum_{k=0}^{n-1} \frac{A^k}{k!}\right] \mathrm{d}t > 0, \tag{19}$$

This gives

$$(R(\gamma))_{i,j} = (-1)^{i+j}\frac{C_{i+j-2}^{i-1}}{\gamma^{i+j-2}}, \quad 1 \le i,\, j \le n. \tag{20}$$

Using (10), we have

$$(T_r'(\gamma)T_r(\gamma))_{i,j} = \frac{(-1)^{i+j}}{\gamma^{i+j-2}} \sum_{k=1}^{n} C_{i-1}^{k-1} C_{j-1}^{k-1}$$
$$= \frac{(-1)^{i+j}}{\gamma^{i+j-2}}\, C_{i+j-2}^{i-1}. \tag{21}$$

This ends the proof of the first item of Lemma 1.
(ii) The (i, j)-th element of the matrix $T_w(\gamma)\mathscr{L}T_w^{-1}(\gamma)$

$$(T_w(\gamma)\mathscr{L}T_w^{-1}(\gamma))_{i,j} = \sum_{k=1}^{n}\sum_{m=1}^{n} \frac{(-1)^{m+j-1}}{\gamma^{k-m}} C_{i-1}^{n-k} C_{n-m}^{j-1} \mathscr{L}_{k,m}. \tag{22}$$

Since $\mathscr{L}$ is a lower triangular matrix then, the summation term is not null if and only if $k \ge m$; which means that, any element of the matrix is a constant or a rational function of the parameter γ. Therefore, by bounding the matrix by its infinity norm, we can always find two constants c_1 and c_2 such that for $\gamma > 1$, we have $\|T_w(\gamma)\mathscr{L}T_w^{-1}(\gamma)\| \le c_1 + \frac{c_2}{\gamma}$.

Using the fact that $(T_r^{-1}(\gamma))_{i,j} = C_{j-1}^{i-1}\,\gamma^{i-1}$ then,

$$(T_r(\gamma)\mathscr{L}T_r^{-1}(\gamma))_{i,j} = \sum_{k=1}^{n}\sum_{m=1}^{n} (-1)^{i+k} C_{k-1}^{i-1} \mathscr{L}_{k,m} C_{j-1}^{m-1} \gamma^{m-k}. \tag{23}$$

Since $\mathscr{L}_{k,m} \ne 0$ for $k \ge m$ then, any element of the matrix $T_r(\gamma)\mathscr{L}T_r^{-1}(\gamma)$ may be a constant or a rational function of the parameter γ. With the same argument, we conclude that for $\gamma > 1$, we can find $c_3 > 0$, $c_4 > 0$ such that

$$\|T_r(\gamma)\mathscr{L}T_r^{-1}(\gamma)\| \le \|T_r(\gamma)\mathscr{L}T_r^{-1}(\gamma)\|_\infty \le c_3 + \frac{c_4}{\gamma}. \quad \square \tag{24}$$

3. Observer-based control

3.1. Main result

This section is devoted to the design of the observer-based controller. By appropriate choice of one parameter we show that a linear-parameterized-observer controller is able to bring the states of the nonlinear time-delay system to the origin whatever the size of the delay. For the sake of simplicity of notation, the time argument is omitted; however, the delayed-state vector $x(t - \tau)$ is noted by x^τ. The complete analysis of the stabilizing controller is given in the following statement.

Theorem 1. *Consider the time-delay nonlinear system (1) under Assumptions 1–2. Let $R(\gamma)$ and $W(\gamma)$ be the solutions of the Lyapunov-like matrix equation (8). Define the nonlinear time-delay observer as*

$$\dot{\hat{x}} = A\,\hat{x} + f(\hat{x}, \hat{x}^\tau) + Bu + \gamma R^{-1}(\gamma)(y - C\,\hat{x}) \tag{25}$$

where $\hat{x}(s) = \hat{\phi}(s)$, $-\tau \le s \le 0$ with $\hat{\phi}(s)$ being any finite known vector. Then, there always exists $\gamma > 1$ that makes system (1) globally asymptotically stable under the observer-based feedback $u = -\gamma B'W^{-1}(\gamma)\,\hat{x}$.

Proof. First, let us prove that there always exists $\gamma > 0$ that makes the observation error $e = \hat{x} - x$ globally asymptotically stable. By composing the difference between system (1) and (25), we get

$$\dot{e} = A\,e + f(\hat{x}, \hat{x}^\tau) - f(x, x^\tau) + \gamma R^{-1}(\gamma)C'(Cx - C\hat{x}). \tag{26}$$

Let $\zeta_\lambda = \hat{x} - \lambda(\hat{x} - x)$, $\eta_\lambda = \hat{x}^\tau - \lambda(\hat{x}^\tau - x^\tau)$ then, by using the Mean-value Theorem, we have

$$\dot{e} = (A - \gamma R^{-1}(\gamma)C'C)e + \int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \zeta}\right|_{\zeta=\zeta_\lambda} (\hat{x} - x)\mathrm{d}\lambda + \int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} (\hat{x}^\tau - x^\tau)\mathrm{d}\lambda. \quad (27)$$

Let $e^\tau = \hat{x}^\tau - x^\tau$, and define

$$Z(e_t) = e'R(\gamma)e + \frac{\gamma}{2}\int_{t-\tau}^{t} e'(s)\, R(\gamma)\, e(s)\, \mathrm{d}s. \quad (28)$$

By differentiation of $Z(e)$, we get

$$\begin{aligned}\dot{Z}(e_t) &= e'(A - \gamma R^{-1}(\gamma)C'C)'R(\gamma)\, e \\ &\quad + e'R(\gamma)(A - \gamma R^{-1}(\gamma)C'C)\, e \\ &\quad + 2e'\, R(\gamma)\int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \zeta}\right|_{\zeta=\zeta_\lambda} e\,\mathrm{d}\lambda \\ &\quad + 2e'\, R(\gamma)\int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} e^\tau\,\mathrm{d}\lambda + \frac{\gamma}{2}e'R(\gamma)\, e \\ &\quad - \frac{\gamma}{2}e'^\tau R(\gamma)\, e^\tau. \end{aligned} \quad (29)$$

Using (8), we can write

$$\dot{Z}(e_t) \le -\frac{\gamma}{2}\, e'R(\gamma)\, e + 2e'\, R(\gamma)\int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \zeta}\right|_{\zeta=\zeta_\lambda} e\,\mathrm{d}\lambda + 2e'\, R(\gamma)\int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} e^\tau\,\mathrm{d}\lambda - \frac{\gamma}{2}e'^\tau R(\gamma)\, e^\tau. \quad (30)$$

Let $z = T_r(\gamma)\, e$, $z^\tau = T_r(\gamma)\, e^\tau$. Since $R(\gamma) = T_r'(\gamma)T_r(\gamma)$ then, we have

$$\dot{Z}(e_t) \le -\frac{\gamma}{2}\, z'z + 2z'\, T_r(\gamma)\int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \zeta}\right|_{\zeta=\zeta_\lambda} T_r^{-1}(\gamma)\, z\,\mathrm{d}\lambda + 2z'\, T_r(\gamma)\int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} T_r^{-1} z^\tau\,\mathrm{d}\lambda - \frac{\gamma}{2}z'^\tau z^\tau. \quad (31)$$

Consequently,

$$\begin{aligned}\dot{Z}(e_t) &\le -\frac{\gamma}{2}\, z'z \\ &\quad + 2\left\| T_r(\gamma)\int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \zeta}\right|_{\zeta=\zeta_\lambda} T_r^{-1}(\gamma)\right\| z'z\,\mathrm{d}\lambda \\ &\quad + 2z'\, T_r(\gamma)\int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} T_r^{-1} z^\tau\,\mathrm{d}\lambda - \frac{\gamma}{2}z'^\tau z^\tau. \end{aligned} \quad (32)$$

Since for given vectors v_1, v_2 and any real matrix $X = X' > 0$ of appropriate dimensions, we have

$$2v_1'v_2 \le v_1'X\, v_1 + v_2'X^{-1}v_2 \quad (33)$$

then,

$$\begin{aligned} &2z'\, T_r(\gamma)\int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} T_r^{-1}(\gamma) z^\tau\,\mathrm{d}\lambda \\ &\le z'Xz + z'^\tau\left(T_r(\gamma)\int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} T_r^{-1}(\gamma)\mathrm{d}\lambda\right)' X^{-1} \\ &\quad\times\left(T_r(\gamma)\int_0^1 \left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} T_r^{-1}\mathrm{d}\lambda\right) z^\tau \\ &\le z'Xz + z'^\tau\int_0^1\left(T_r(\gamma)\left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} T_r^{-1}(\gamma)\right)' X^{-1} \\ &\quad\times\left(T_r(\gamma)\left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} T_r^{-1}(\gamma)\right) z^\tau\,\mathrm{d}\lambda. \end{aligned} \quad (34)$$

By taking $X = \frac{\gamma}{4}I$, we have

$$\begin{aligned}\dot{Z}(e_t) &\le -\frac{\gamma}{4}\, z'z \\ &\quad + 2\left\| T_r(\gamma)\left.\frac{\partial f(\zeta,\eta)}{\partial \zeta}\right|_{\zeta=\zeta_\lambda} T_r^{-1}(\gamma)\right\| z'z \\ &\quad + \frac{4}{\gamma}\left\| T_r(\gamma)\left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} T_r^{-1}(\gamma)\right\|^2 z'^\tau z^\tau - \frac{\gamma}{2}z'^\tau z^\tau. \end{aligned} \quad (35)$$

Taking into account that the Jacobian matrices are lower triangular then, by using the result of Lemma 1, we can find a set of positive constants α, β, α_τ, and β_τ, that are independent of γ, such that for $\gamma > 1$

$$\begin{aligned} &\left\| T_r(\gamma)\left.\frac{\partial f(\zeta,\eta)}{\partial \zeta}\right|_{\zeta=\zeta_\lambda} T_r^{-1}(\gamma)\right\| \le \alpha + \frac{\beta}{\gamma} \\ &\left\| T_r(\gamma)\left.\frac{\partial f(\zeta,\eta)}{\partial \eta}\right|_{\eta=\eta_\lambda} T_r^{-1}(\gamma)\right\| \le \alpha_\tau + \frac{\beta_\tau}{\gamma}. \end{aligned} \quad (36)$$

This implies that

$$\dot{Z}(e_t) \le \left(-\frac{\gamma}{4} + 2\frac{\beta}{\gamma} + 2\alpha\right) z'z + \left(-\frac{\gamma}{2} + \frac{4}{\gamma}\left(\alpha_\tau + \frac{\beta_\tau}{\gamma}\right)^2\right) z'^\tau z^\tau. \quad (37)$$

By choosing $\gamma > 1$ such that $-\frac{\gamma}{4} + 2\frac{\beta}{\gamma} + 2\alpha < 0$ and $-\frac{\gamma}{2} + \frac{4}{\gamma}(\alpha_\tau + \frac{\beta_\tau}{\gamma})^2 < 0$ then, we conclude that the observation error is globally asymptotically stable.

It is sufficient now to prove that the closed-loop system under the observer-based controller is input-to-state-stable with respect to the observation error. The closed-loop dynamics are given by

$$\dot{x} = A\, x + f(x, x^\tau) - \gamma\, BB'W^{-1}(\gamma)(x + e). \quad (38)$$

Let

$$V(x_t) = x'W^{-1}(\gamma)x + \frac{\gamma}{2}\int_{t-\tau}^{t} x'(s)\, W^{-1}(\gamma)\, x(s)\,\mathrm{d}s. \quad (39)$$

Then, the time-derivative of $V(x_t)$ along the trajectories of system (38) is given by

$$\begin{aligned}\dot{V}(x_t) &= x'[A - \gamma\, BB'W^{-1}(\gamma)]'W^{-1}(\gamma)x \\ &\quad + x'W^{-1}(\gamma)[A - \gamma\, BB'W^{-1}(\gamma)]\, x \\ &\quad - 2\,\gamma x'W^{-1}(\gamma)BB'W^{-1}(\gamma)\, e + 2\, x'W^{-1}(\gamma)f(x, x^\tau) \\ &\quad + \frac{\gamma}{2}x'W^{-1}(\gamma)\, x - \frac{\gamma}{2}x'^\tau W^{-1}(\gamma)\, x^\tau. \end{aligned} \quad (40)$$

Using (8), we have

$$\begin{aligned}\dot{V}(x_t) &= x'\left[-\frac{\gamma}{2}\, W^{-1}(\gamma) - \gamma\, W^{-1}(\gamma)BB'W^{-1}(\gamma)\right] x \\ &\quad - 2\,\gamma x'W^{-1}(\gamma)BB'W^{-1}(\gamma)\, e + 2\, x'W^{-1}(\gamma)f(x, x^\tau) \\ &\quad - \frac{\gamma}{2}x'^\tau W^{-1}(\gamma)\, x^\tau. \end{aligned} \quad (41)$$

By taking the following change of state variable: $\xi = T_w(\gamma)\,x$, $\xi^\tau = T_w(\gamma)\,x^\tau$, we can then bound $\dot V$ as follows

$$\dot V(x_t) \le -\frac{\gamma}{2}\xi'\xi - 2\gamma\xi'(T_w(\gamma)BB'T_w'(\gamma))T_w(\gamma)\,e + 2\,\xi'T_w(\gamma)f(T_w^{-1}(\gamma)\,\xi, T_w^{-1}(\gamma)\,\xi^\tau) - \frac{\gamma}{2}\xi'^\tau\xi^\tau. \quad (42)$$

Since $\|T_w(\gamma)BB'T_w'(\gamma)\| \le n$ then,

$$\dot V(x_t) \le -\frac{\gamma}{2}\xi'\xi + 2\,n\gamma\,\|\xi\|\,\|T_w(\gamma)\,e\| + 2\,\xi'T_w(\gamma)f(T_w^{-1}(\gamma)\,\xi, T_w^{-1}(\gamma)\,\xi^\tau) - \frac{\gamma}{2}\xi'^\tau\xi^\tau. \quad (43)$$

Since for given $a \in \mathbb{R}$, $b \in \mathbb{R}$ and for all $\epsilon > 0$, we have

$$2a\,b \le \epsilon\,a^2 + \epsilon^{-1}\,b^2. \quad (44)$$

Then, by taking $\epsilon = \frac{1}{n\gamma}$, we can write that

$$2\,n\gamma\,\|\xi\|\,\|T_w(\gamma)\,e\| \le \|\xi\|^2 + (n\gamma)^2 e'W^{-1}(\gamma)e. \quad (45)$$

This gives

$$\dot V(x_t) \le -\frac{1}{2}(\gamma-2)\xi'\xi + (n\gamma)^2 e'W^{-1}(\gamma)e + 2\,\xi'T_w(\gamma)f(T_w^{-1}(\gamma)\,\xi, T_w^{-1}(\gamma)\,\xi^\tau) - \frac{\gamma}{2}\xi'^\tau\xi^\tau. \quad (46)$$

Let $p_\lambda = \lambda T_w^{-1}(\gamma)\,\xi$, $q_\lambda = \lambda\,T_w^{-1}(\gamma)\xi^\tau$. Since $f(0,0) = 0$ then, we have

$$f(T_w^{-1}(\gamma)\,\xi, T_w^{-1}(\gamma)\,\xi^\tau) - f(0,0) = \int_0^1 \left.\frac{\partial f(p,q)}{\partial p}\right|_{\substack{p=p_\lambda\\ q=q_\lambda}} T_w^{-1}(\gamma)\,\xi\,\mathrm{d}\lambda + \int_0^1 \left.\frac{\partial f(p,q)}{\partial q}\right|_{\substack{p=p_\lambda\\ q=q_\lambda}} T_w^{-1}(\gamma)\,\xi^\tau\,\mathrm{d}\lambda. \quad (47)$$

Define $\mathscr{L}_\lambda \triangleq \frac{\partial f(p,q)}{\partial p}|_{\substack{p=p_\lambda\\ q=q_\lambda}}$, $\mathscr{L}_\lambda^\tau \triangleq \frac{\partial f(p,q)}{\partial q}|_{\substack{p=p_\lambda\\ q=q_\lambda}}$. Then,

$$\dot V(x_t) \le -\frac{1}{2}(\gamma-2)\xi'\xi + (n\gamma)^2 e'W^{-1}(\gamma)e + 2\int_0^1 \xi'T_w(\gamma)\mathscr{L}_\lambda T_w^{-1}(\gamma)\,\xi\,\mathrm{d}\lambda + 2\int_0^1 \xi'T_w(\gamma)\mathscr{L}_\lambda^\tau T_w^{-1}(\gamma)\,\xi^\tau\,\mathrm{d}\lambda - \frac{\gamma}{2}\xi'^\tau\xi^\tau. \quad (48)$$

Using the fact that $\mathscr{L}_\lambda$ and $\mathscr{L}_\lambda^\tau$ are lower triangular matrices with bounded entries then, with $\gamma > 1$ we can always find two positive constants α and β such that

$$2\int_0^1 \xi'T_w(\gamma)\mathscr{L}_\lambda T_w^{-1}(\gamma)\,\xi\,\mathrm{d}\lambda \le 2\int_0^1 \|T_w(\gamma)\mathscr{L}_\lambda T_w^{-1}(\gamma)\|\xi'\xi\,\mathrm{d}\lambda = 2\left(\alpha + \frac{\beta}{\gamma}\right)\xi'\xi. \quad (49)$$

On the other hand, we have for any $\varepsilon > 0$

$$2\int_0^1 \xi'T_w(\gamma)\mathscr{L}_\lambda^\tau T_w^{-1}(\gamma)\,\xi^\tau\,\mathrm{d}\lambda \le \varepsilon^{-1}\int_0^1 \xi'\xi\mathrm{d}\lambda + \varepsilon\int_0^1 \|T_w(\gamma)\mathscr{L}_\lambda^\tau T_w^{-1}(\gamma)\|^2\xi'^\tau\xi^\tau\,\mathrm{d}\lambda. \quad (50)$$

Let α_τ, β_τ be positive constants verifying the following inequality for $\gamma > 1$

$$\|T_w(\gamma)\mathscr{L}_\lambda^\tau T_w^{-1}(\gamma)\|^2 \le \left(\alpha_\tau + \frac{\beta_\tau}{\gamma}\right)^2. \quad (51)$$

This gives

$$\dot V(x_t) \le \left[-\frac{\gamma}{2} + 2\frac{\beta}{\gamma} + 2\alpha + \varepsilon^{-1} + 1\right]\xi'\xi + \left[-\frac{\gamma}{2} + \varepsilon\left(\alpha_\tau + \frac{\beta_\tau}{\gamma}\right)^2\right]\xi'^\tau\xi^\tau + (n\gamma)^2 e'W^{-1}(\gamma)e. \quad (52)$$

By choosing $\gamma > 1$ such that

$$\begin{cases} -\dfrac{\gamma}{2} + 2\dfrac{\beta}{\gamma} + 2\alpha + \varepsilon^{-1} + 1 = -\pi_1^2, \\ -\dfrac{\gamma}{2} + \varepsilon\left(\alpha_\tau + \dfrac{\beta_\tau}{\gamma}\right)^2 = -\pi_2^2, \end{cases} \quad (53)$$

Then, $\dot V(x_t) \le -\pi_1^2\,\xi'\xi - \pi_2^2\,\xi'^\tau\xi^\tau + (n\gamma)^2 e'W^{-1}(\gamma)\,e$. Since the observation error e vanishes to zero when time elapses then, $\dot V(x_t)$ becomes decreasing after a transient period of time which implies the asymptotic stability of ξ. This ends the proof. □

3.2. Discussion

Filtering and observer-based control of both linear and nonlinear time-delay systems has been treated with different approaches including convex optimization methods, (Fridman & Shaked, 2002; Ibrir, Xie, & Su, 2005; Ivănescu, Dion, Dugard, & Niculescu, 2000; Kwon, Park, & Lee, 2006; Xu & Dooren, 2002; Zemouche & Boutayeb, 2009) frequency domain techniques, pole placement methods and Lyapunov designs (Fairman & Kumar, 1986). Convex-optimization techniques or Linear-Matrix-Inequality procedures have shown their powerfulness in handling multi-objective tasks for broad classes of systems (Boyd, Ghaoui, Feron, & Balakrishnan, 1994; Ibrir et al., 2006; Moon, Park, Kwon, & Lee, 2001; Niculescu, 2001; Pertew, Marquez, & Zhao, 2007; Scherer, Gahinet, & Chilali, 1997; Wang et al., 2002; Zhang, Polycarpou, & Parisini, 2010). However, the numerical tractability of the solutions are generally stated as sufficient conditions that do not always hold. Although necessary and sufficient conditions for the stabilization of linear time-delay systems can be formulated as linear matrix inequalities, see for instance Xu and Lam (2005), the output-feedback problem is fundamentally stated as a non-convex optimization problem even for non-delayed systems, see e.g., Song and Hedrick (2004), Ibrir (2006), Ibrir and Diop (2008), Kwon et al. (2006) and Lien (2004). For systems in strict-feedback form the results are numerous, see for example Krishnamurthy and Khorrami (2007) and the references therein.

The existence of an explicit solution to the output-feedback problem for Lipschitz nonlinear time-delay systems simplifies enormously the analysis and the computation of an observer-based controller. In the present paper, the developed state-feedback controller is not explicitly dependent on the system nonlinearities but quite dependent on the upper bounds of the nonlinearity Jacobian matrices. This fact makes the overall parametrization realizable with only one parameter "γ". It is important to mention that even the controller does not depend on the size of the delay, the information of the delay remains necessary to construct the state estimates. This difficulty in such a control exercise may be removed by changing the structure of the observer, i.e., making the observer independent from the information of the time delay.

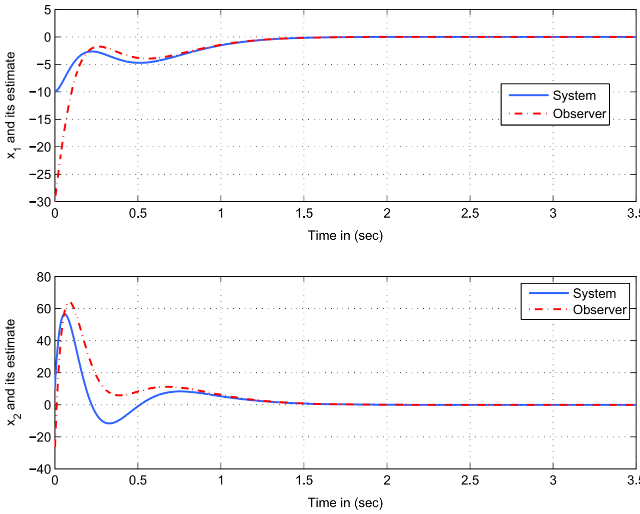

Fig. 1. The performance of the observer-based controller for $\tau = 0.5$ s.

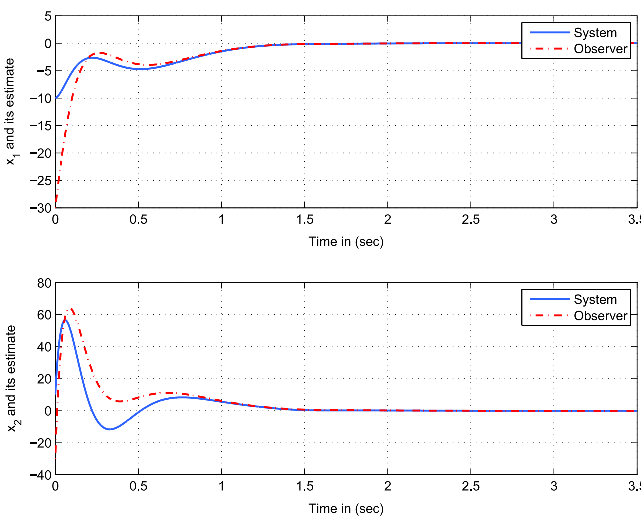

Fig. 2. The performance of the observer-based controller for $\tau = 1$ s.

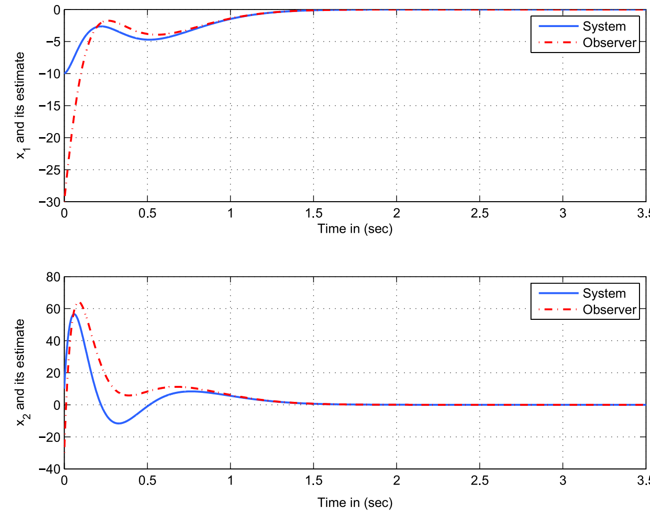

Fig. 3. The performance of the observer-based controller for $\tau = 4$ s.

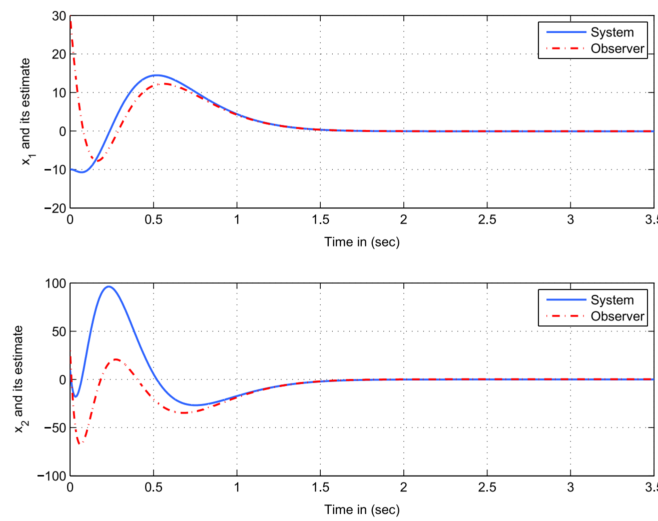

Fig. 4. The performance of the observer-based controller for $\tau = 10$ s.

In this case, even the delayless observer cannot recover the exact estimates when used alone, it shall be able to guarantee the convergence of the system to the origin when combined with a stabilizing controller.

Remark 1. It is obvious that the result remains valid in the case where the system nonlinearity is also dependent on time and globally Lipschitz with respect to the state vectors x and x^{τ}, respectively.

4. Simulation results

Consider the time-delay system

$$\dot{x}_1 = x_2(t) - \frac{1}{2}\frac{x_1(t)}{1+x_1^2(t)},$$
$$\dot{x}_2 = \frac{1}{2}\arctan(x_2(t-\tau)) + \frac{1}{2}\frac{x_2(t-\tau)}{1+x_2^2(t-\tau)} + u(t),$$
$$y(t) = x_1(t), \tag{54}$$

where τ is supposed to be constant and $\phi(t) = 0$ for $-\tau \le t < 0$, $x_1(0) = -10$, $x_2(0) = 10$. The system nonlinearities are globally Lipschitz which allows to set $\alpha = 1$, $\alpha_\tau = 1$ and $\beta = \beta_\tau = 0$. By taking $\epsilon = 2$ then, for $\gamma > 7$ inequalities (53) are verified.

Fig. 1 shows the performance of the observer-controller for a constant delay equal to 0.5 s. For $\gamma = 7.1$ the stabilizing controller is able to steer all the states of the time-delay system when the amount of the delay is increased to 1 and 4 s as shown in Figs. 2 and 3. In Fig. 4, we show that for a large delay $\tau = 10$ s and for different initial conditions the convergence of the system states to the origin is still verified with the same gain which means that the parameters of the observer-based controller are not dependent on the size of the delay.

5. Conclusion

Irrespective of the size of the delay, we showed that the global asymptotic stability of time-delay systems, written in triangular form, can be achieved by using a linear observer-based feedback. As a result, linear time-delay systems having triangular structures are also stabilizable with the same feedback. Consequently, nonlinear systems having triangular structures and involving multiple time delays are also stabilized with linear parameterized feedback. However, the design in this case requires higher gains to dominate the effects of the lower triangular matrices induced by multiple-time delays.

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