ON THE CONTROL OF NONHOLONOMIC SYSTEMS IN POWER FORM.

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ABSTRACT

In this paper, we develop discontinuous controllers for stabilizing $n-$dimensional nonholonomic mechanical systems given in power form. We show that the stabilization of the whole system turns out to be a simultaneous stabilization of $(n-3)$ driftless subsystems. The control strategy is based on the technique of invariant manifolds. Moreover, it is shown that we could improve the rate of convergence of the state vector by the use of dynamic controllers. A numerical procedure is proposed to filter the noisy measurement without affecting the dynamic of the controller.

Keywords: Invariant manifolds, Stabilization, Numerical methods.

1. INTRODUCTION

The stabilization of driftless mechanical systems has received more attention during the last few years. Such systems arise when modelling mechanical systems with nonholonomic constraints, i.e, with non-integrable constraints. The examples are numerous, and the reader is reffered to [8], [21] for recent works in modelling and control of such type of systems or the existance of change of coordinates that permit to put the original systems in their power form.

In practice, it is often required to maintain the mechanical system around a desired configuration. This task is viewed as a stabilization problem where the desired configuration is made an asymptotically stable equilibrium point. Several solutions were proposed to solve this problem. The first approach is an open-loop strategy. This involves defining the control inputs as functions of time so that the initial state of the model is transferred to the desired final state [17]. By the application of such a strategy of control, it is clear that the system performances are degraded by modelling additive external disturbances. The second approach consists in feeding back the state of the system by stabilizing controllers which ensure a certain robustness to modelling errors and noisy measurement. In [5], Brockett has shown that driftless systems cannot be asymptotically stabilized around any desired point with continuous autonomous feedback. This result has motivated researchers to derive a diversity of discontinuous controllers. In [4] Bloch et al derived piecewise analytic feedbacks to achieve stability. Canudas de Wit and Sϕrdalen developed peicewise smooth controllers for a set of low dimensional examples [9] , while Somson demonstrated that continuous timeperiodic feedbacks could stabilize a nonholonomic system [19] . Coron showed that for a large class of driftless systems there exists a smooth time periodic feedback that renders the desired equilibrium point globally asymptotically stable [6]. In [18] Pomet gave the methodology to adapt the ideas in Coron's proof to explicit an algorithm for deriving time-periodic smooth feedback for a more restrictive class of driftless systems. Explicit expressions for the stabilization of a chained form of driftless systems were proposed by Teel et al in [20]. Other authors like Fliess et all [10], Martin and Rouchon [14], have looked at the stabilization of such a type of systems via differential flatness approach.

Invariant manifolds techniques appeared to be powerful tools for the stabilization of driftless mechanical systems. The interested reader is referred to [21] to see the progress in this area.

In this article, we develop a new kind of discontinuous controllers in conjunction with invariant manifolds. We shall show that when the states lie on the intersection of the set of invariant manifolds, dynamic controllers are proposed to improve the rate of convergence of the whole states. It is shown that by the application of these dynamic controllers, we obtain the same invariant surfaces as obtained by linear static controllers. Moreover, the methodology of the design of the control laws described herein, is thus adapted to any driftless system which could not put in power form. In addition, the rate of the convergence of the states is mastered by a suitable choice of regulation models (first or second order model) to the first two states of the system. In order to give a good insight of this paper, in section 2 we show how we construct the invariant manifolds and give the detailed strategy of the stabilizing controller to the $n-$dimensional systems in power form. Section 3 is devoted to the numerical procedure used as a filter. In section 4 we give the stabilizing controller for a particular case of a driftless system in power form where simulations results are presented.

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2. STATEMENT OF THE PROBLEM

2.1. CONSTRUCTION OF THE INVARIANT MANIFOLDS

We consider the $n-$dimensional underactuated systems in power form with two inputs and two integrators described by the state model equation

$$(\Sigma)\begin{cases} \dot{x}_1 = u_1, \\ \dot{x}_j = \frac{1}{(j-2)!}{x_1}^{j-2}u_2, \qquad j = 2,\cdots,n, \\ \dot{u}_1 = v_1, \\ \dot{u}_2 = v_2. \end{cases}$$

For $\beta > 0$, let us define

$$v_1 \;:=\; -2\beta\, u_1 - 4\,\beta^2\, x_1, \tag{1}$$
$$v_2 \;:=\; -2\beta\, u_2 - 4\,\beta^2\, x_2. \tag{2}$$

Then, we cite the following result

Lemma 2.1 *Consider the system (Σ) under the feedback (1), (2) with the initial conditions $u_1(0) = -\beta\, x_1(0)$, $u_2(0) = -\beta\, x_2(0)$*

- *Then, the family of the manifolds*

$$s_j(x) := \{x_{j+2} - \frac{1}{(j+1)!}\,{x_1}^j\, x_2 = 0, \quad j = 1,\cdots,n-2\},$$

  are invariant surfaces which verify $\dot{s}(x) = 0$.

- *For all*

$$x_0 \in \bigcap_{j=1}^{n-2}\{x_{j+2} - \frac{1}{(j+1)!}\,{x_1}^j\, x_2 = 0, \quad j = 1,\cdots,n-2\},$$

  all the states converge exponentially to zero. □

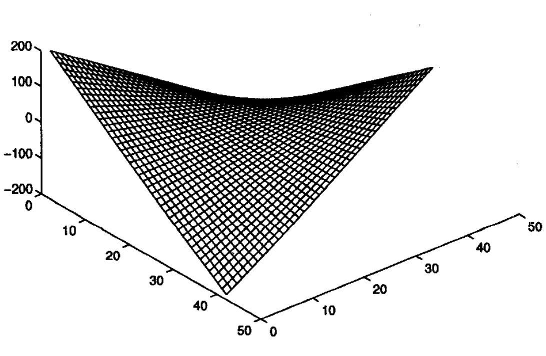

Figure 1: The resulted invariant manifold $s_1(x) := x_3 - \frac{1}{2}x_1 x_2$ for the $n-$dimensional system in power form.

Proof. By the application of the dynamic controllers (1), (2), the time responses of the states $x_1(t)$ and $x_2(t)$ are respectively

$$x_1(t) \;=\; x_1(0)\, e^{-\beta t}\, \cos(\sqrt{3}\beta t),$$
$$x_2(t) \;=\; x_2(0)\, e^{-\beta t}\, \cos(\sqrt{3}\beta t).$$

Let $\eta_0 = \frac{x_1(0)}{x_2(0)}$, then $\frac{x_1(t)}{x_2(t)} = \eta_0$, thus $u_1 = \eta_0\, u_2$. Consequently

$$x_1\, u_2 = x_2\, u_1. \tag{3}$$

For $j = 1,\cdots,n-2$

$$\begin{aligned} \dot{s}_j &= \dot{x}_{j+2} - \frac{j}{(j+1)!}{x_1}^{j-1}\, x_2\, u_1 - \frac{1}{(j+1)!}{x_1}^{j}\, u_2 \\ &= \frac{j}{(j+1)!}{x_1}^{j-1}(x_1\, u_2 - x_2\, u_1) \\ &= 0. \end{aligned}$$

Integrating both sides of the model equations (Σ) over $[0,\ t]$ then the time responses of the states $x_k(t)$, $k = 1,\cdots,n$ have the following forms

$$\begin{cases} x_1(t) = x_1(0)\, e^{-\beta\, t}\, \cos(\sqrt{3}\beta\, t), \\ x_2(t) = x_2(0)\, e^{-\beta\, t}\, \cos(\sqrt{3}\beta\, t), \\ x_k(t) = \underbrace{\int \frac{1}{(k-2)!}{x_1}^{k-2}(\tau)\, u_2(\tau)\, d\tau}_{\psi_k(t)} + \\ \underbrace{x_k(0) - \frac{1}{(k-1)!}\, {x_1}^{k-2}(0)\, x_2(0)}_{s_k(0)}, \qquad k = 3,\cdots,n. \end{cases}$$

Substituting

$$x_1(t) \;=\; x_1(0)\, e^{-\beta t}\, \cos(\sqrt{3}\beta t),$$
$$u_2(t) \;=\; -\beta x_2(0)\, e^{-\beta t} \cos(\sqrt{3}\beta t) - \sqrt{3}\beta x_2(0)\, \sin(\sqrt{3}\beta t),$$

in the expression of the k-th state $x_k(t)$ and using the fact that for any $\zeta_1 > 0$

$$\begin{aligned} &\lim_{\tau\to\infty} \int e^{-\zeta_1 \tau}\, \cos(\zeta_2 \tau)\, d\tau = \\ &\lim_{\tau\to\infty} e^{-\zeta_1 \tau}\, \left[(\zeta_2\, \sin(\zeta_2 \tau) - \zeta_1\, \cos(\zeta_2 \tau))/({\zeta_1}^2 + {\zeta_2}^2)\right] \\ &= 0. \end{aligned}$$

and since $\cos^\gamma t$, $(\gamma \in \mathbb{N}^*)$ could be factorized as follows

$$\frac{1}{2^{\gamma-1}} \times \sum_{\substack{i\in\{\gamma,\gamma-2,\cdots,1\} \\ j\in\{0,1,\cdots,\frac{\gamma-1}{2}\}}} C_\gamma^{\gamma-j} \cos(i\, t), \quad \text{if } \gamma \text{ is od,}$$

$$\frac{1}{2^{\gamma-1}} \times \sum_{\substack{i\in\{\gamma,\gamma-2,\cdots,2\} \\ j\in\{0,1,\cdots,\frac{\gamma}{2}-1\}}} C_\gamma^{\gamma-j} \cos(i\, t) + \frac{C_\gamma^{\frac{\gamma}{2}}}{2\gamma}, \quad \text{if } \gamma \text{ is even.}$$

Thus, one could conclude that $\lim_{t\to\infty}(\psi_k(t),\ k = 3,\cdots,n) = 0$. If the vector x_0 is chosen as $(s_k(x_0) = 0,\ k = 3,\cdots,n)$ then all the state trajectories $(x_1(t), x_2(t),\cdots,x_n(t))$ converge exponentially to the origin. ◊

Remark 2.1 *If we applied the static controllers $u_1 = -x_1$, $u_2 = -x_2$ instead of the dynamic controllers (1), (2), we would obtain the same invariant manifolds $(s_j(x),\ j = 1,\cdots,n-2)$ defined in lemma 2.1.* •

Remark 2.2 *The manifolds* $(s_j(x),\ j=1,\cdots,n-2)$ *become no longer invariant if the coefficient* β *is not the same for the controllers* v_1 *and* v_2.•

Remark 2.3 *The dynamic controllers (1), (2) do not impose a constraint of observation of the velocities, because these ones represent the control inputs.*•

2.2. THE STABILIZATION PROCESS

The question now is how to steer the system (Σ) to the origin if the initial states do not comply with the condition $\cap_{j=1}^{n-2} s_j(x_0) = 0$?. The key to realize such control design is to find a controller which renders the conditions $\left\{\cap_{j=1}^{n-2} s_j(x)=0\right\}$ verified, and then switch on the dynamic controller (1), (2) which ensures the invariance of the manifolds $(s_j(x),\ j=1,\cdots,n-2)$. Let

$$\begin{aligned} v_{1,1} &= u_1, \\ v_{2,1} &= x_1 u_2 - x_2 u_1, \\ \xi_{1,1} &= x_1, \\ \xi_{2,1} &= s_1, \\ \xi_{3,1} &= s_2, \\ &\vdots \\ \xi_{j,1} &= s_{j-1}, \\ &\vdots \\ \xi_{n-1,1} &= s_{n-2}. \end{aligned}$$

be the first change of coordinates, and suppose that there exist $v_{1,1} = v^\star{}_{1,1}$, and $v_{2,1} = v^\star{}_{2,1}$ that stabilize the following $(n-1)$-dimensional system

$$(\Sigma)_{\xi_1} \begin{cases} \dot{\xi}_{1,1} = v_{1,1}, \\ \dot{\xi}_{j,1} = \frac{j-1}{j!}\, \xi_{1,1}{}^{j-2}\, v_{2,1}, \quad j=2,n-1. \end{cases} \tag{4}$$

at the origin. It is obvious that by switching on the dynamic controllers (1), (2) we will stabilize the system (Σ) at the origin. We conclude that the stabilization of the n-dimentional system requires the stabilization of another driftless system of dimension $n-1$. Therefore, we could repeat the same process of control for the obtained $(n-1)-$dimensional system by constructing new hypersurfaces under the effect of static or dynamic controllers and designing a new subsystem of lower dimension after a change of coordinates. This process must be stopped until we get a 3-dimensional driftless system. The stabilization process will be started from the last resulting system and ends with the dynamic controllers (1), (2). Exploiting the structure of the resulting system $(\Sigma)_{\xi_1}$, then we arrive to the main result summarized in the following statement

Theorem 2.1 *Consider the system* (Σ) *under the feedback (1), (2) and let*

$$\begin{cases} v_{1,\ell} := v_{1,\ell-1}, \\ v_{2,\ell} := \xi_{1,\ell-1}\, v_{2,\ell-1} - \ell!\, \xi_{2,\ell-1}\, v_{1,\ell-1}, \\ \xi_{1,\ell} := \xi_{1,\ell-1}, \\ \xi_{j,\ell} := s^{[\ell-1]}{}_{j-1}; \\ j=2,\cdots,n-\ell,\ \ell=1,\cdots,n-3. \end{cases} \tag{5}$$

be the familly of the recursive change of coordinates for each step ℓ *such that*

$$\begin{cases} v_{1,0} := u_1, \\ v_{2,0} := u_2, \\ \xi_{j,0} := x_j,\ j=1,\cdots,n, \\ s^{[0]}{}_j(.) := s_j(.),\ j=1,\cdots,n-2. \end{cases} \tag{6}$$

For each step $\ell,\ \ell \in \{1,2,\cdots,n-3\}$, *we construct the subsystem* (Σ_{ξ_ℓ}) *whose states are* $\xi_\ell = (\xi_{1,\ell},\cdots,\xi_{n-\ell,\ell})$. *Under the static feedback*

$$\begin{cases} v_{1,\ell} = -\xi_{1,\ell}, \\ v_{2,\ell} = -(\ell+1)!\, \xi_{2,\ell}, \end{cases} \tag{7}$$

or the dynamic feedback

$$\begin{cases} \dot{v}_{1,\ell} = -2\beta\, v_{1,\ell} - 4\beta^2\, \xi_{1,\ell};\ \beta > 1, \\ \dot{v}_{2,\ell} = -(\ell+1)!\,(2\beta\, v_{2,\ell} + 4\beta^2\, \xi_{2,\ell}), \end{cases} \tag{8}$$

with

$$\begin{cases} v_{1,\ell}(0) = -\beta\, \xi_{1,\ell}(0) \\ v_{2,\ell}(0) = -\beta\, \xi_{2,\ell}(0). \end{cases}$$

- *The next* $\ell-$*th generated subsystem* (Σ_{ξ_ℓ}) *has the following form*

$$\begin{cases} \dot{\xi}_{1,\ell} = v_{1,\ell}, \\ \dot{\xi}_{j,\ell} = \frac{(j-1)}{(j+\ell-1)!}\, \xi^{j-2}{}_{1,\ell}\, v_{2,\ell},\ j=2,\cdots,n-\ell. \end{cases} \tag{9}$$

- *For each subsystem* (Σ_{ξ_ℓ}) *the manifolds*

$$s^{[\ell]}{}_j(\xi_\ell) := \{\xi_{j+2,\ell} - \frac{(\ell+1)!}{(j+\ell+1)!}\, \xi^j{}_{1,\ell}\, \xi_{2,\ell} = 0, \quad j=1,\cdots,n-\ell-2\}, \tag{10}$$

are invariant hypersurfaces. △

Proposition 2.1 *Consider the last generated system* $(\Sigma)_{\xi_{n-3}}$ *of dimension 3 described by the state model equations*

$$\begin{cases} \dot{\xi}_{1,n-3} = v_{1,n-3}, \\ \dot{\xi}_{2,n-3} = \frac{1}{(n-2)!}\, v_{2,n-3}, \\ \dot{\xi}_{3,n-3} = \frac{2}{(n-1)!}\, \xi_{1,n-3}\, v_{2,n-3}. \end{cases} \tag{11}$$

Let $\xi^\star{}_{1,n-3} := \frac{(n-1)}{2}\frac{\xi_{3,n-3}(0)}{\xi_{2,n-3}(0)} \neq 0$. *For all* $\xi_{n-3} \in \mathbb{R}^3$ *such that* $\xi_{2,n-3}(0) \neq 0$, *then the discontinuous controller*

$$\begin{cases} v_{1,n-3} = -k_2 sign(\xi_{1,n-3} - \xi^\star{}_{1,n-3}),\ v_{2,n-3} = 0, \\ \quad \text{if } \xi_{1,n-3} \neq \xi^\star{}_{1,n-3}, \\ v_{2,n-3} = -k_1 (n-2)!\, \xi_{2,n-3},\ v_{1,n-3} = 0, \\ \quad \text{if } \xi_{1,n-3} = \xi^\star{}_{1,n-3}. \end{cases} \tag{12}$$

ensures the asymptotic convergence of the states $(\xi_{2,n-3}, \xi_{3,n-3})$ *to the origin.* □

Proof. If we apply the controllers

$$\begin{cases} v_{1,n-3} = 0, \\ v_{2,n-3} = -k_1(n-2)!\,\xi_{2,n-3}, \quad k_1 > 0. \end{cases} \tag{13}$$

Then the state $\xi_{2,n-3}$ converges to zero with the rate k_1 while $\xi_{3,n-3}$ converges to the constant value

$$\xi_{3,n-3}(0) - \frac{2}{(n-1)}\,\xi_{1,n-3}(0)\,\xi_{2,n-3}(0).$$

It is clear that if we maintain $v_{2,n-3}$ equal to zero and act on the controller $v_{1,n-3}$ such that the state $\xi_{1,n-3}$ reach the value $\frac{(n-1)}{2}\frac{\xi_{3,n-3}(0)}{\xi_{2,n-3}(0)}$, then by switching on the controller (10) both the states $\xi_{2,n-3}$ and $\xi_{3,n-3}$ converge exponentially to the origin with the rate k_1. ◊

After the stabilization of the states $(\xi_{2,n-3}, \xi_{3,n-3})$ at the origin by the controllers (12), we continue the stabilization process for the subsystems $((\Sigma)_{\xi_{n-4}}, (\Sigma)_{\xi_{n-5}}, \cdots, (\Sigma)_{\xi_1})$ by switching at each step on the controller (6). This process should end by switching on the controllers (1), (2).

Remark 2.4 *At the end of the stabilization of any system $(\Sigma)_{\xi_\ell}$, the controllers $v_{2,\ell}$ might be singular if $x_1 = 0$. We could leave these singularities by applying the controllers $v_{1,\ell} = c$, $c > 0$, $v_{2,\ell} = 0$ for a short period $T > 0$. These controllers preserve the invariance of the surfaces $\xi_{j,\ell}(.)$, $j = 2, \cdots, n-\ell$.* •

3. FILTERING THE MEASUREMENTS

In this section we show how we filter the noisy measurement by a numerical procedure. The details of the latter will not be reported here. The reader is referred to [12] for more explanations about the algorithm. In the meantime, we restrict ourselves to give an idea how the procedure should work. Let us suppose that the measurements are collected at a regular step of time and let $(y_1, \cdots, y_n)^t$ be the mooving vector of the noisy data which corresponds to the equally spaced instants $(t_1, t_2, \cdots, t_n)$. The algorithm is based on the fact that the random measurement error ϵ verifies the following conditions

$$E[\epsilon_i] = 0,$$
$$E[\epsilon_i \epsilon_j] = 0, \qquad i \neq j,$$
$$E\left[{\epsilon_i}^2\right] = \sigma^2.$$

where σ^2 denotes the variance of noise. Our strategy of filtering is to consider the following constrained optimization problem

$$\min \sum_{i=m}^{n-1} \left[\hat{y}_i^{(m)}\,(\Delta t)^m\right]^2. \tag{14}$$

subject to the constraint

$$\sum_{i=1}^{n} \left[\frac{\hat{y}(t_i) - y(t_i)}{\delta y_i}\right]^2 \leq n\,\sigma^2, \qquad \hat{y} \in C^{(m)}[t_1, t_n]. \tag{15}$$

The notation $\hat{y}^{(m)}$ denotes the m-th derivative (in the sense of finite differences) of the function $\hat{y}$, δy_i, $i = 1, \cdots, n$ are positive numbers taken as estimates of the standard deviation in y_i. Δt designates the regular forward difference of t, equal to $t_{i+1} - t_i$. We seek the minimum of the criterion in the space of the B-spline functions. We replace then $\hat{y}$ by the B-spline of order $2m$, i.e,

$$\sum_{i=1}^{2m} \alpha_i b_{i,2m}(t), \tag{16}$$

such that $\alpha = (\alpha_i,\ i = 1 \cdots, 2m) \in \mathbb{R}^{2m}$, and $b_{i,2m}$ is the i-th positive B-spline function. The solution of this constrained optimization problem turns out to be the coefficients of the control vector α. We give

$$\alpha = B^{-1}\left(y - D^2\,T^t\,u(\lambda)\right) \tag{17}$$

such that λ is the Lagrange parameter which minimizes

$$\|D\,T^t\,u(\lambda)\|^2.$$

$\|.\|$ is the Euclidean norm and T is an $(m \times 2m)$ matrix of a general row

$$(-1)^{m+j-1}\frac{m!}{(j-1)!(m-j+1)!}, \quad j = 1, \cdots, m+1, \tag{18}$$

$D^{-2} = diag({\delta y_1}^{-2}, \cdots, {\delta y_n}^{-2})$ and B is an $n \times n$ matrix such that $B_{i,j} := b_{j,2m}(t_i)$. u is an $m \times 1$ vector equal to $(T\,D^2\,T^t + \lambda\,I)^{-1}\,T\,y$. I designates the $n \times n$ identity matrix.

4. SIMULATION RESULTS

Consider the 4-dimensional driftless system in power form

$$(\Sigma_4)\begin{cases} \dot{x}_1 = u_1, \\ \dot{x}_2 = u_2, \\ \dot{x}_3 = x_1\,u_2, \\ \dot{x}_4 = \frac{1}{2}{x_1}^2\,u_2, \\ \dot{u}_1 = v_1, \\ \dot{u}_2 = v_2. \end{cases} \tag{19}$$

with the initial condition $x_0 = (1\,1\,1\,\frac{1}{2}\,0\,0)$. We suppose that the whole state are measured with a random error ϵ of variance $\sigma^2 = 0.1$. Under the effect of the controller (1), (2) where $\beta = 1$, the resulting invariant surfaces are

$$s_1(x) \;=\; x_3 - \frac{1}{2}x_1\,x_2,$$
$$s_2(x) \;=\; x_4 - \frac{1}{6}{x_1}^2\,x_2.$$

As we have mentioned in section 2, the system $(\Sigma)_{\xi_1}$ takes the following form

$$\begin{cases} \dot{\xi}_{1,1} = v_{1,1}, \\ \dot{\xi}_{2,1} = \frac{1}{2}\,v_{2,1}, \\ \dot{\xi}_{3,1} = \frac{1}{3}\,\xi_{1,1}\,v_{2,1}. \end{cases} \tag{20}$$

For all x_0 such that $s_1(x_0) \neq 0$, let $x^\star = \frac{3}{2}\frac{s_2(0)}{s_1(0)} \neq 0$, then the controllers

$$\begin{cases} u_1 = -k_2\,\text{sign}(x_1 - x^\star);\ v_{2,1} = 0, \\ \qquad \text{if}\, x_1 \neq x^\star;\ \text{and}\ k_2 > 0, \\ u_2 = -2k_1\,S_1(x)/x_1;\ u_1 = 0, \\ \qquad \text{if}\, x_1 = x^\star, \\ u_1 = \int_{t_s}^{t} -2u_1(\tau) - 4x_1(\tau)\,d\tau + u_1(t_s) \\ \qquad \text{if}\, s_1(x) = s_2(x) = 0, \\ u_2 = \int_{t_s}^{t} -2u_2(\tau) - 4x_2(\tau)\,d\tau + u_2(t_s) \\ \qquad \text{if}\, s_1(x) = s_2(x) = 0. \end{cases} \tag{21}$$

stabilize the sytem (Σ_4) at the origin. t_s is the first instant where $s_1(t_s) = s_2(t_s) = 0$. $u_1(t_s)$ and $u_1(t_s)$ have to satisfy the condition

$$\begin{cases} u_1(t_s) = -x_1(t_s), \\ u_2(t_s) = -x_2(t_s). \end{cases}$$

5. CONCLUSIONS

In this paper we have presented another methodology of design of discontinuous stabilizing controllers for $n-$dimensional driftless systems in power form. Our control strategy was based on the construction of invariant manifolds obtained by the integration of the system equation with respect to dynamic controllers. The generalization of such a control method can also be applied to $n-$dimensional driftless system of any form.

6. REFERENCES

[1] A. Astolfi, 1994, "On the control of nonholmonic systems". In proceedings of the 33rd IEEE CDC, 3481–3486.

[2] A. Bloch and S. Drakunov, 1994, "Stabilization of nonholmonic system via sliding mode". In proceedings of the 33rd IEEE CDC, 2961–2963.

[3] A. M. Bloch and N. H. McClamroch, 1989, "Control of mechanical syetms with classical nonholnomic constraints". In proceedings of the 28th IEEE CDC, 201–205.

[4] A. M. Bloch, M. Reyhanoglu, and N. H. McClamroch, 1992, "Control and stabilization of nonholonomic dynamic systems". IEEE trans. on Auto. Contr., 37, 11, 1746–1757.

[5] R. W. Brockett, 1983, "Asymptotic stability and feedback stabilization". In R. W. Brockett, R. S. Millman and H. J. Sussman, editors, (Differential Geometric Control theory), 181–191, Birkhauser.

[6] J.-M. Coron, 1992, "Global asymptotic stabilization for controllable systems without drift". Mathematics of Control Signals and Systems, 5, 295–312.

[7] C. C. de Wit, 1991, "Exponential stabilization of mobile robots with nonholmonic constraints". In proceedings of the 30th IEEE CDC, 692–697, December.

[8] C. C. de Wit, 1997, "Invariant manifolds: a tool for stabilization". In the procedings of the International Workshop on Modelling and Control of Mechanical Systems, 1, 253–275.

[9] C. C. de Wit and O. J. Sϕrdalen, 1992, "Examples of peicewise smooth stabilization of driftless nl systems with less inputs than states". IFAC (NOLCOS), 26–30.

[10] M. Fliess, J. Levine, P. Martin, and P. Rouchon, 1995, "Design of trajectory stabilizing feedback for driftless flat systems". In the procedings of the 3rd ECC, 1882–1887.

[11] S. Ibrir, 1997, "Numerical observer controller for the stabilization of a nonminimum phase mechanical system". In proceedings of the International Workshop on Modelling and Control of Mechanical Systems , 2. Imperial college, London.

[12] S. Ibrir 1998, "A numerical algorithm for filtering and state observation". accepted for publication in proceedings of IEE UKACC, England.

[13] Z.-P. Jiang and H. Nijmeijer, 1997, "Backstepping-based tracking control of nonholonomic chained systems". In proceedings (CDROM) of the 4th ECC, Brussels.

[14] P. Martin and P. Rouchon, 1994, "Any (controllable) driftless system with 3 inputs and 5 states is flat". In proceedings of the 33rd IEEE CDC, 4038–4042, December.

[15] P. Morin, J.-B. Pomet, and C. Samson, 1996, "Design of homogenous time-varing stabilizing control laws for driftless controllable systems via oscillatory approximation of lie brackets in closed-loop". Rapport de Recherche N^0 : 3077 , INRIA Sophia Antipolis, December.

[16] P. Morin and C. Samson, 1994, "Time-varing feedback stabilization of the attitude of a rigid spacecraft with two controls". Rapport de Recherche N^0 : 2275 , INRIA Sophia Antipolis.

[17] R. Murray and S. S. Sastry, 1993, "Nonholmonic motion palnning: Steering using sinusoids". IEEE Trans. on Auto. Control, 38, 700–716.

[18] J.-B. Pomet, 1992, "Explicit design of time-varing stabilization control laws for a class of controllable systems without drift". Systems & Control Letters, 18, 2, 147–158.

[19] C. Samson, 1992, "Velocity and torque feedback control of a nonholonomic cart". advanced robot control, Springer Verlag.

[20] A. R. Teel, R. M. Murray, and G. Walsh, 1992, "Nonholnomic control systems: From steering to stabilization with sinusoids". In proceedings of the IEEE CDC, 1603–1609.

[21] P. Tsiotras, 1997, "Invariant manifold techniques for control of underactuated mechanical systems". In the procedings of the International Workshop on Modelling and Control of Mechanical Systems, 1, 277–292.

[22] P. Tsiotras and J. M. Longuski, 1994, "Spin-axis stabilization of symmetric spacecraft with two torques". Systems & Control Letters, 23, 395–402, December.

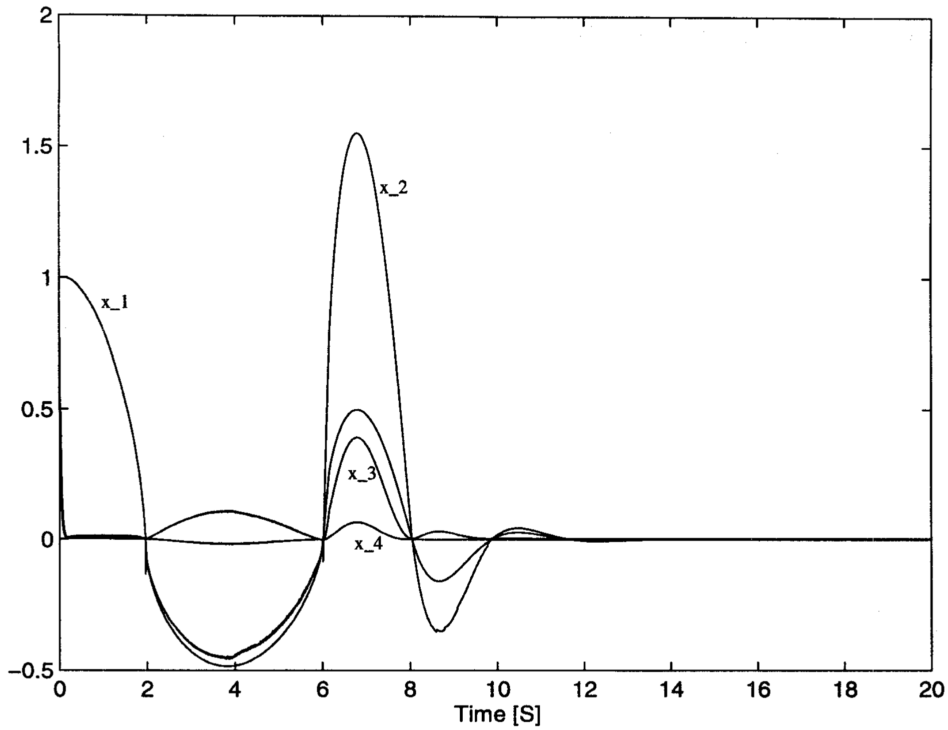

Figure 2: The states x_1, x_2, x_3, and x_4.

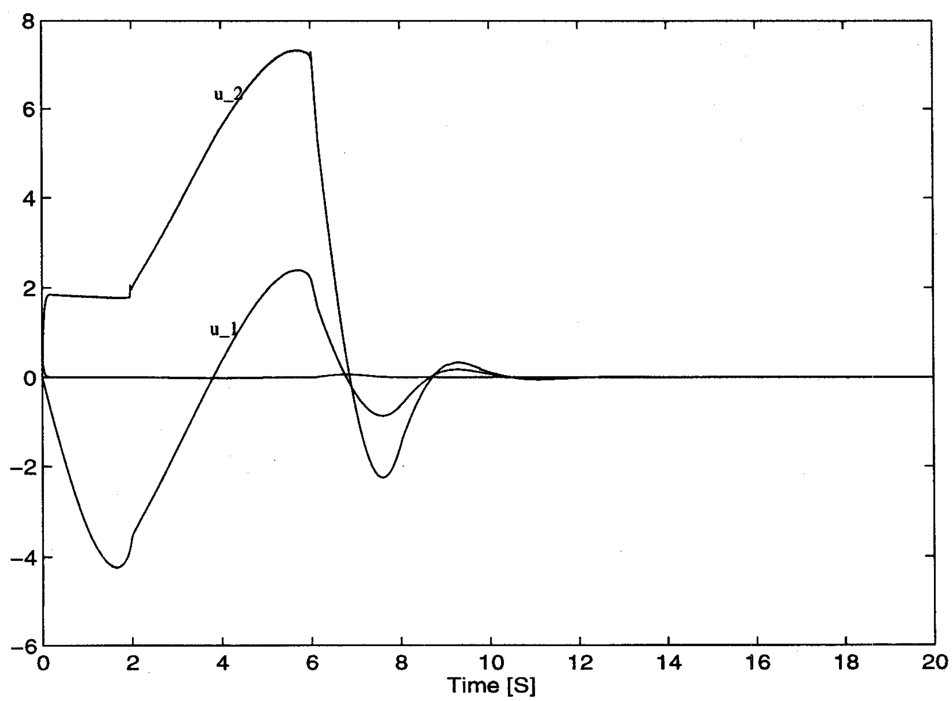

Figure 3: The controllers u_1 and u_2.