New parameterized hysteresis model and stabilization challenge of a class of hysteresis input systems

Salim Ibrir, Chun-Yi Su and Wen Fang Xie

***Abstract*— In this paper, a new rate-dependent hysteresis model is developed. Subsequently, the basic properties of the developed model are detailed. Trajectory tracking of a class of nonlinear systems that involve such kind of hysteresis is then discussed without any approximation or inversion of the hysteresis model. Illustrative example is given to show the novelty of the theoretical results.**

***Index Terms*— Hysteresis modelling; adaptive control; trajectory tracking; mechatronics.**

I. INTRODUCTION

Modelling and control of hysteresis have been attracting the attentions of many researchers all over the world. This is due to the fact that numerous emerging actuation devices and materials exhibit strong hysteresis characteristics during their operation. Examples of practical systems include high precision positioning systems, piezoelectric transducers, chaotic oscillators, electromagnetic machines and lightweight space structures [1], [2], [3]. Due to the importance of hysteresis modelling, a revival of interest has been devoted to this subject since the last decade. Significant contributions to hysteresis modelling have been reported in references [4], [2], [1]. These results have found large applications in control engineering, especially with the development of mechatronic systems that created a clear need for dealing with hysteresis nonlinearities, see e.g., [5], [6], [7], [8], [9], [10] and the references therein.

In this paper, we propose a new rate-dependent hysteresis model that involves the higher derivatives of the control input. By the variation of the developed hysteresis parameters, various shapes of saturated hysteresis phenomena are formed. The developed dynamical model is ideal for describing ferromagnetic hysteresis phenomena and electro-magnetic actuators based on hysteresis operators. There are at least two main advantages to use the developed model. First, the dynamics of the hysteresis model can be easily associated to the system model being considered. Consequently, controller design, which is the main concern of the paper, becomes straightforward. Second, by extension of the hysteresis parameters to those of the system under consideration, the observation and identification procedures shall be simplified. Through a numerical example, we show that for tracking exercises, we are not in need of inverting the dynamics of hysteresis to extract the information of the control input. As a result, a smooth controller is sufficient to make the system states track prescribed reference trajectories. Throughout this paper, $\mathbb{R}$, and $\mathbb{Z}$ stand for the set of real and integer numbers, respectively. The notation A' stands for the matrix transpose of A. I_n is the $n \times n$ identity matrix. $\dot{x}(t)$ denotes the usual differentiation of the real valued vector $x(t)$ with respect to time. $r^{(i)}(t)$ is the i-th time-derivative of the scalar function $r(t)$. $\mathrm{col}_n(A)$ stands for the n-th column of the matrix A. $\delta_{i,j}$ is the kronecker operator defined as $\delta_{i,j} = \begin{cases} 1 & \text{if}, i = j \\ 0 & \text{otherwise.} \end{cases}$

The authors are with the department of mechanical & industrial engineering, Concordia university, Department of Mechanical & Industrial Engineering. 1455, de Maisonneuve Blv. Montreal, Canada H3G 1M8. E-mails: $\{\mathrm{sibrir, cysu, wfxie}\}$@encs.concordia.ca.

II. NEW HYSTERESIS DYNAMICAL MODEL

A. Control problem

Consider the n-dimensional nonlinear system

$$\begin{aligned} \dot{x}_i(t) &= f_i(x_{i+1}(t)) = x_{i+1}(t),\ 1 \le i \le n-1 \\ \dot{x}_n(t) &= f_n(x(t), h(t), \theta) = \sum_{i=1}^{\mu} \theta_i \varphi_i(x(t)) + h(t), \end{aligned} \tag{1}$$

where $x(t) : [0\ \infty) \longmapsto \mathbb{R}^n$ is the state vector, $u(t) : [0\ \infty) \longmapsto \mathbb{R}$ is a scalar control input, $(\theta_i)_{1 \le i \le \mu}$ are constant unknown parameters, and $(\varphi_i(x(t)))_{1 \le i \le \mu}$ are at least $\mathscr{C}^{(1)}$ functions verifying $\varphi_i(0) = 0$. $h(t)$ is the hysteresis output. The presence of unknown hysteresis inputs in existing practical systems arises several major problems in control of such systems. First, the model parameters of well-known hysteresis are generally unknown and the control input is not extractable, which makes the design of stabilizing controllers an extremely difficult task. Second, the need of modelling the rate-dependent hysteresis behaviors in control exercises motivates the research in this area.

B. The hysteresis model

In this section, we introduce a new class of hysteresis systems modelled by the following differential equations

$$\begin{aligned} \dot{\xi}(t) &= \psi(\xi(t), u(t))\, g(u(t), \dot{u}(t)) + \ell(u(t), \dot{u}(t), \ddot{u}(t)), \\ y_h(t) &= H(\xi(t), u(t)), \end{aligned} \tag{2}$$

where $\xi(t) : [0\ \infty) \longmapsto \mathbb{R}^n$ is the system continuous states, $u(t) : [0\ \infty) \longmapsto \mathbb{R}$ is the piecewise $\mathscr{C}^{(2)}$ input signal and $y_h(t) : [0\ \infty) \longmapsto \mathbb{R}$ is the hysteresis output. $\psi : \mathbb{R}^n \times \mathbb{R} \longmapsto \mathbb{R}^{n \times m}$ is continuous and locally Lipschitz and $g(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \longmapsto \mathbb{R}^m$ is continuous and $\mathscr{C}^{(1)}$. Notice that if $\ell(u(t), \dot{u}(t), \ddot{u}(t)) \equiv 0$ and $g(\cdot,\cdot)$ depends only upon $\dot{u}(t)$, then (2) coincides with the generalized Duhem model [1]. The model we are going to present establishes a mapping from an input u and a real number h_0 to an output function $h(\cdot,\cdot)$ which solves (2). We characterize this mapping by the hysteresis operator $\mathscr{H}(u, h_0)$. Before presenting the new dynamical hysteresis model and its properties, let us give the following definitions that are useful for the rest of the paper.

Definition 1 (Rate dependence): The hysteresis operator $\mathscr{H}(u, h_0)$ is said rate-dependent if the path of the couple $(u, \mathscr{H}(u, h_0))$ is dependent to increasing time homeomorphism.

Definition 2 (Rate independence): The hysteresis operator is rate-independent if the output of the hysteresis is invariant with respect to changes of time scales.

Now, we are ready to present the form of $\psi(\cdot,\cdot)$ and $g(\cdot,\cdot)$ in order to describe a certain class of saturated hysteric phenomena. It will be shown later that this new model is suitable for stabilization and tracking exercises, and it does not need to be inverted to extract the information of the control input. For a given control input $u(t) : [0\ \infty) \longmapsto \mathbb{R}$, we define the hysteresis output $h(t) : [0\ \infty) \longmapsto \mathbb{R}$ as the solution of the following differential equation

$$\dot{h}(t) = \Big(-p_1\, h^q(t) + p_2 \arctan(u(t))\Big) \left|\frac{\dot{u}(t)}{1+u^2(t)}\right| + p_3 \frac{\dot{u}(t)}{1+u^2(t)} + p_4 \frac{\ddot{u}(t)}{(1+u^2(t))^2},\ h(0) = h_0. \tag{3}$$

where p_1, p_2, p_3 are positive constant unknown parameters, p_4 is a real unknown parameter and q is a predefined odd positive integer. Here $n = 1$, $y_h(t) = h(t) = \xi(t)$, $\psi(\xi(t), u(t)) = -p_1\, \xi^q(t) + p_2 \arctan(u(t))$, and $g(u(t), \dot{u}(t)) = \left|\frac{\dot{u}(t)}{1+u^2(t)}\right|$.

The new rate-dependent differential model (3) has two useful characteristics. The possibility of controller design without model inversion reveals as the main characteristics. The second important characteristic is the linear parameterization (except q) property of the hysteresis model (3) that plays a key role in the design of adaptive compensation strategies. The proposed hysteresis model (3) can be either rate-dependent or rate-independent according to the value of p_4. For this reason let us introduce the following statements.

Proposition 1: For all $p_4 \neq 0$ the hysteresis operator $\mathscr{H}(u, h_0)$ is rate-dependent.

Proof: Let $\pi(t)$ be a positive time scale such that $\pi(0) = 0$. Let $h_\pi(t) = h(\pi(t))$ and $u_\pi(t) = u(\pi(t))$. Since $\dot{\pi} > 0$, then we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} h_\pi(t) + \Big(p_1\, h_\pi^q(t) - p_2 \arctan(u_\pi(t))\Big) \left|\frac{\frac{\mathrm{d}}{\mathrm{d}t} u_\pi(t)}{1+u_\pi^2(t)}\right| \\
&- p_3 \frac{\frac{\mathrm{d}}{\mathrm{d}t} u_\pi(t)}{1+u_\pi^2(t)} - p_4 \frac{\frac{\mathrm{d}^2}{\mathrm{d}t^2} u_\pi(t)}{(1+u_\pi^2(t))^2} \\
&= \Bigg[\frac{\mathrm{d}}{\mathrm{d}\pi} h_\pi(t) + \Big(p_1\, h_\pi^q(t) - p_2 \arctan(u_\pi(t))\Big) \left|\frac{\frac{\mathrm{d}}{\mathrm{d}\pi} u_\pi(t)}{1+u_\pi^2(t)}\right| \\
&- p_3 \frac{\frac{\mathrm{d}}{\mathrm{d}\pi} u_\pi(t)}{1+u_\pi^2(t)}\Bigg] \dot{\pi} - p_4 \frac{\frac{\mathrm{d}^2}{\mathrm{d}t^2} u_\pi(t)}{(1+u_\pi^2(t))^2} \\
&\neq 0.
\end{aligned}$$

This ends the proof.

When $p_4 = 0$ then the dynamics equation $\frac{\mathrm{d}}{\mathrm{d}\pi} h_\pi(t) + \Big(p_1\, h_\pi^q(t) - p_2 \arctan(u_\pi(t))\Big) \left|\frac{\frac{\mathrm{d}}{\mathrm{d}\pi} u_\pi(t)}{1+u_\pi^2(t)}\right| - p_3 \frac{\frac{\mathrm{d}}{\mathrm{d}\pi} u_\pi(t)}{1+u_\pi^2(t)} = 0$, is verified. and consequently, $\frac{\mathrm{d}}{\mathrm{d}t} h_\pi(t) + \Big(p_1\, h_\pi^q(t) - p_2 \arctan(u_\pi(t))\Big) \left|\frac{\frac{\mathrm{d}}{\mathrm{d}t} u_\pi(t)}{1+u_\pi^2(t)}\right| - p_3 \frac{\frac{\mathrm{d}}{\mathrm{d}t} u_\pi(t)}{1+u_\pi^2(t)} = 0$. which immediately implies that the hysteresis operator is rate-independent.

III. Boundedness of the hysteresis output

A. Boundedness property

In this section we study the conditions under which the hysteresis output is bounded. Similar to other hysteresis models as the so-called Bouc-Wen model [11], a general explicit solution of the hysteresis output (3) is not available. In this paper, we restrict ourselves to detail the boundedness property of the hysteresis (3) for $q = 1$. This breakdown is given in the following statement.

Lemma 1: Consider the input-output hysteresis model

$$\dot{h}(t) = \Big(-p_1\, h(t) + p_2 \arctan(u(t))\Big) \left|\frac{\dot{u}(t)}{1+u^2(t)}\right| + p_3 \frac{\dot{u}(t)}{1+u^2(t)} + p_4 \frac{\ddot{u}(t)}{(1+u^2(t))^2}. \tag{4}$$

Then for any piecewise $\mathscr{C}^{(2)}$ input $u(t) : [0\ \infty) \longmapsto \mathbb{R}$, the output $h(t) : [0\ \infty) \longmapsto \mathbb{R}$ is globally bounded by

$$\sup_{t\geq 0} |h(t)| \leq \frac{\pi}{2}\frac{p_2}{p_1} + \frac{p_2}{p_1^2} + \frac{p_3}{p_1} + p_4\Big(2 + \frac{1}{p_1}\Big) \sup_{t\geq 0} \left|\frac{\dot{u}(t)}{1+u^2(t)}\right| + c\, e^{\frac{\pi}{2} p_1}, \tag{5}$$

where c is a real constant that depends on the initial conditions $h(0)$, $u(0)$, $\dot{u}(0)$ and $\ddot{u}(0)$.

Proof: The proof is omitted due to space limitation.

From result of lemma 1, we conclude that the upper bound of the hysteresis output depends basically on both the parameters $(p_i)_{1\leq i\leq 4}$ and the rate of changes in the applied input $u(t)$. For bounded inputs, the ratios $\frac{p_2}{p_1}$, $\frac{p_2}{p_1^2}$, $\frac{p_3}{p_1}$, and $\frac{p_4}{p_1}$ affect considerably the upper bound that can reach the hysteresis output. When the parameter $p_4 = 0$, the upper and the lower bounds of the hysteresis output will not depend on the applied input $u(t)$.

The differential model (3) has the property to be BIBO (bounded-input bounded-output). Actually, the bounded-output property is always verified for any continuously differentiable input $u(t)$ where $\lim_{t\to\infty} u(t)$ exists. From result of lemma 1, the upper bound of the output does not depend only on the hysteresis parameters, but it is quite dependent upon the rate of changes of $u(t)$, namely $\frac{\dot{u}(t)}{1+u^2(t)}$. The boundedness properties of the hysteresis model (3) is summarized in the following statements.

Proposition 2: The hysteresis system (3) is BIBO (bounded-input-bounded-output) for all piecewise $\mathscr{C}^2$ input $u(t)$ satisfying $\sup_{t\geq 0} |\dot{u}(t)| < \infty$.

Proof: From result of Lemma 1, we see that the upper and the lower bound of $h(t)$ depends on $\frac{\dot{u}(t)}{1+u^2(t)}$ which is globally bounded under the assumption $\sup_{t\geq 0} |\dot{u}(t)| < \infty$ which is the claim.

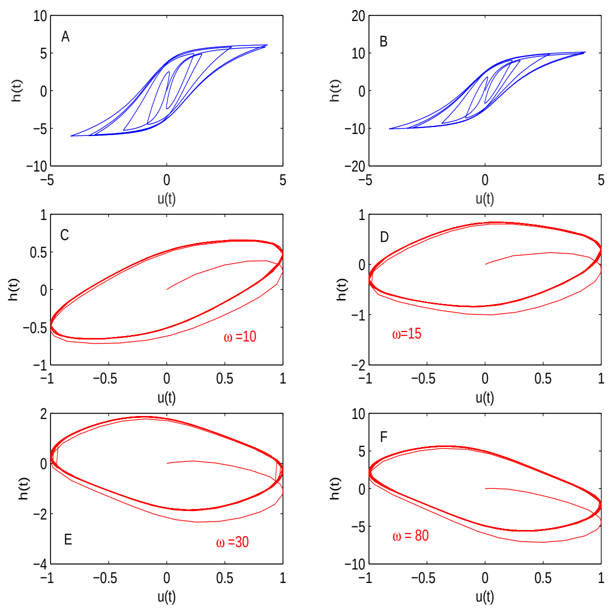

Fig. 1. Parameters and input frequency effects

Proposition 3: Consider system (3). If $u(t)$ is piecewise $\mathscr{C}^{(2)}$ and $\lim_{t\to\infty} u(t)$ exists, then the output $h(t)$ is globally bounded.

Proof: Since

$$\int_0^\infty \frac{\dot{u}(\tau)}{1+u^2(\tau)}\,\mathrm{d}\,\tau = \arctan(u(\infty)) - \arctan(u(0)) = \text{Const.}, \tag{6}$$

then by the use of Barbala Lemma, we can say that $\lim_{t\to\infty} \frac{\dot{u}(t)}{1+u^2(t)} = 0$. Under the assumption that $u(t)$ is piecewise $\mathscr{C}^{(2)}$ input over $[0\ \infty)$, then we conclude immediately that $\frac{\dot{u}(t)}{1+u^2(t)}$ is globally bounded and hence, the solution $h(t)$ is globally bounded. This ends the proof.

B. Hysteresis parameters effects

- In subplot A of Fig. 1, the minor loops of the hysteresis are clearly shown for the following hysteresis parameters $p_1 = 2$, $p_2 = 2.7$, $p_3 = 10$ $p_4 = 0.8$ and $u(t) = 2.2\sin(\frac{\pi}{3}t) + 2.2\sin(\frac{\sqrt{2}\pi}{3}\,t)$. In subplot B of Fig. 1 the hysteresis parameters are changed to $p_1 = 2$, $p_2 = 6.7$, $p_3 = 15$, $p_4 = 0.8$, $q = 1$ which means that the values of the ratios $\frac{p_3}{p_1}$ and $\frac{p_2}{p_1}$ has augmented and consequently, the upper bounds of the hysteresis output have been increased. In the remaining subplots C, D, E, and F of Fig. 1, the responses of the hysteresis for a sinusoid input $u(t) = \sin(\omega\, t)$ of different frequencies are represented. By setting $p_1 = 1$, $p_2 = 1.5$, $p_3 = 1$, $p_4 = 0.2$, $q = 1$ and increasing the frequency of the sinusoid, the form of the hysteresis characteristic changes correspondingly.

IV. Passivity and Sense of Circulation

The system $(u(t), \dot{h}(t))$ is passive for $p_4 < 0$. This property is characterized in terms of an integral inequality condition given by the following statement.

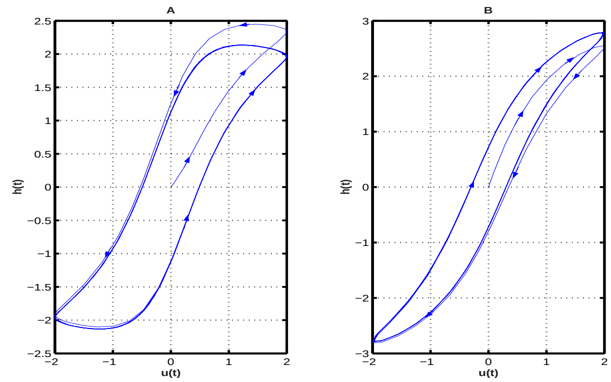

Fig. 2. The sense of circulation

Proposition 4: For $p_4 \leq 0$, and for any bounded triplet $(u(t), \dot{u}(t), h(t))$ satisfying the input-output differential model (3) and for all $\delta > 0$, there exists a real α such that

$$\int_0^\delta \dot{h}(\tau)\, u(\tau)\, \mathrm{d}\tau \geq \alpha. \tag{7}$$

Proof: The proof is omitted here due to space limitation.

Depending on the sign of p_4, the circulation may be counter-clockwise or clockwise. Since the passivity property implies the counterclockwise circulation, see [12]. Then we conclude that when $p_4 < 0$, the circulation becomes counter-clockwise. In Fig. 2 we show that the sense of circulation may be clockwise for $p_4 > 0$. As a result, the parameter p_4 does not only affect the width of the hysteresis, but it decides on the sense of the circulation. A hysteresis with counter-clockwise loops is referred to as a passive hysteresis, see for example [13], [12], [14]. The passivity property can be also seen from the ability of hysteresis to absorb energy coming from input signals and dissipate only a finite amount of energy. In Fig. 2, we have represented the sense of circulation for two different values of p_4. In the subplot A, the hysteresis parameters are $p_1 = 1$, $p_2 = 2.5$, $p_3 = 1$, $q = 1$, $p_4 = -1$. For the input signal $u(t) = 2\sin(3t)$, the hysteresis is crossed counterclockwise. Alternatively, in the subplot B, the circulation becomes clockwise for the hysteresis parameters $p_1 = 1$, $p_2 = 2.5$, $p_3 = 3$, $q = 1$, $p_4 = 1$ and the same control input.

V. Output Hysteresis Estimation by Synchronization

In this section, we show that the hysteresis output can be measured after a finite converging time if the hysteresis parameters are a priori identified. Suppose now that all the parameters of the hysteresis model are known. Then, the hysteresis output $h(t)$ is the solution of the following differential equation

$$\dot{h}(t) = \Big(-p_1\, h^q(t) + p_2 \arctan(u(t))\Big)\left|\frac{\dot{u}(t)}{1+u^2(t)}\right| + p_3\frac{\dot{u}(t)}{1+u^2(t)} + p_4\frac{\ddot{u}(t)}{(1+u^2(t))^2}, \tag{8}$$

where the real initial condition $h(0) = h_0$ is assumed unknown. For any differentiable control input $u(t) \in \mathbb{R}$ such that $\dot{u}(t) \neq 0$, the estimated hysteresis output $\hat{h}(t)$, with $\hat{h}(0) = \hat{h}_0$, is readily constructed as a copy of (8), that is

$$\dot{\hat{h}}(t) = \Big(-p_1\,\hat{h}^q(t) + p_2 \arctan(u(t))\Big)\left|\frac{\dot{u}(t)}{1+u^2(t)}\right| + p_3\frac{\dot{u}(t)}{1+u^2(t)} + p_4\frac{\ddot{u}(t)}{(1+u^2(t))^2}. \tag{9}$$

Denoting $e_h(t) = \hat{h}(t) - h(t)$, we obtain

$$\dot{e}_h = -p_1\left|\frac{\dot{u}(t)}{1+u^2(t)}\right|\big(\hat{h}^q(t) - h^q(t)\big). \tag{10}$$

By the mean-value Theorem, the dynamics of the error $e_h(t)$ is rewritten as follows

$$\dot{e}_h = -p_1\,q\left|\frac{\dot{u}(t)}{1+u^2(t)}\right|\times \int_0^1 \Big(\hat{h}(t) - \lambda(\hat{h}(t)-h(t))\Big)^{q-1}\mathrm{d}\,\lambda\; e_h(t) \tag{11}$$

By taking the Lyapunov function $V(e_h) = \frac{1}{2}e_h^2$, then the first derivative of $V(e_h)$ along the trajectories of (11) is

$$\dot{V}(e_h) = -p_1\,q\,e_h(t)\left|\frac{\dot{u}(t)}{1+u^2(t)}\right|\times \int_0^1 \Big(\hat{h}(t) - \lambda(\hat{h}(t)-h(t))\Big)^{q-1}\mathrm{d}\,\lambda\; e_h(t) \tag{12}$$

Since q is an odd integer number, then the quantity

$$\Big(\hat{h}(t) - \lambda(\hat{h}(t)-h(t))\Big)^{q-1} \geq 0, \quad 0 \leq \lambda \leq 1. \tag{13}$$

This immediately implies that $\dot{V}(e_h) \leq 0$. Consequently $\hat{h}(t)$ will converge in finite time to the real hysteresis output $h(t)$ whatever $\hat{h}_0$.

VI. Trajectory tracking

In this section, we consider the problem of adaptive trajectory tracking of a class of nonlinear systems that involve hysteresis inputs of form (3). A dynamical system in Brunovski form with scalar hysteresis input (3) possesses the generalized controller form initiated in reference [15]. This fact can be considered as an additional and useful property that simplifies the complexity of controller design. Consider system (1) and assume that $h(t)$ is defined as in section II. In order to complete the description of system (1), the following assumptions are considered.

Assumption 1: The state vector $x(t)$ is available for feedback.

Assumption 2: The hysteresis output $h(t)$ is available for feedback.

By augmenting system (1) with the state $x_{n+1}(t) = h(t)$. and defining $A_n \in \mathbb{R}^{n\times n}$, $B_n \in \mathbb{R}^n$, $p \in \mathbb{R}^3$, $\theta \in \mathbb{R}^\mu$, $\varphi(x(t)) \in \mathbb{R}^\mu$, and $\psi(x(t), u(t), \dot{u}(t)) \in \mathbb{R}^3$ as

$$\begin{aligned} A_n &:= [A_{i,j}]_{1\leq i,j\leq n} := [\delta_{i,j-1}]_{1\leq i,j\leq n}\,, \\ B_{n,\omega} &:= [B_i]_{1\leq i\leq n} := [\delta_{i,\omega}]_{1\leq i\leq n}\,, \\ \theta &:= \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_\mu \end{bmatrix}', \\ p &:= \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}', \\ \varphi(x(t)) &:= \begin{bmatrix} \varphi_1(x(t)) & \cdots & \varphi_\mu(x(t)) \end{bmatrix}, \\ \psi(x_{n+1}(t), u(t), \dot{u}(t)) &:= \left[-x_{n+1}^q(t)\left|\frac{\dot{u}(t)}{1+u^2(t)}\right| \quad \arctan(u(t))\left|\frac{\dot{u}(t)}{1+u^2(t))}\right| \quad \frac{\dot{u}(t)}{1+u^2(t)}\right]. \end{aligned} \tag{14}$$

Then the dynamics of system (1) is rewritten as

$$\dot{x}(t) = A_{n+1}\,x(t) + B_{n+1,n}\varphi(x(t))\,\theta + B_{n+1,n+1}\psi(\cdot,\cdot,\cdot)\,p + p_4\,B_{n+1,n+1}\frac{\ddot{u}(t)}{(1+u^2(t))^2}. \tag{15}$$

The objective is to design an adaptive controller such that $\lim_{t\to\infty} x_i(t) - r^{(i-1)}(t) = 0$, $1 \leq i \leq n$ where the reference $r = r(t) : [0\ \infty) \longmapsto \mathbb{R}$ and its higher derivatives are bounded. The tracking design is summarized in the following statement.

Theorem 1: Consider system (15) under the feedback

$$\begin{aligned} \dot{\zeta}_1(t) &= \zeta_2(t), \\ \dot{\zeta}_2(t) &= -\hat{\rho}(t)(1+\zeta_1^2(t))^2\Big(\mathrm{col}_{n+1}'(P_\gamma^{-1})\,s(t) \\ &\quad + \psi(x_{n+1}(t), \zeta_1(t), \zeta_2(t))\hat{p} \\ &\quad + \sum_{i=1}^{\mu}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_n(t)}\varphi(x(t))\hat{\theta}(t) \\ &\quad + \sum_{i=1}^{\mu}\sum_{j=1}^{n}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_j(t)}x_{j+1}(t) \\ &\quad - r^{(n+1)}(t) + \varphi(x(t))\varphi'(x(t))\,\mathrm{col}_n'(P_\gamma^{-1})s(t) \\ &\quad + \varphi(x(t))\varphi'(x(t))\sum_{i=1}^{\mu}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_n(t)}\times \\ &\quad \mathrm{col}_{n+1}'(P_\gamma^{-1})s(t)\Big), \\ u(t) &= \zeta_1(t), \end{aligned} \tag{16}$$

$$
\begin{aligned}
\dot{\hat{\theta}}(t) &= \varphi'(x(t))\,\mathrm{col}_n^{'}(P_\gamma^{-1})s(t)\\
&\quad + \varphi'(x(t))\sum_{i=1}^{\mu}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_n(t)}\mathrm{col}_{n+1}^{'}(P_\gamma^{-1})s(t),\\
\dot{\hat{p}}(t) &= \psi'(x_{n+1}(t),u(t),\dot{u}(t))\,\mathrm{col}_{n+1}^{'}(P_\gamma^{-1})s(t),\\
\dot{\hat{\rho}}(t) &= s'(t)\,\mathrm{col}_{n+1}(P_\gamma^{-1})\,\mathrm{col}_{n+1}^{'}(P_\gamma^{-1})s(t)\\
&\quad + \hat{p}'\psi'(x_{n+1}(t),u(t),\dot{u}(t))\,\mathrm{col}_{n+1}^{'}(P_\gamma^{-1})s(t)\\
&\quad - r^{(n+1)}(t)\,\mathrm{col}_{n+1}^{'}(P_\gamma^{-1})s(t)\\
&\quad + s'(t)\,\mathrm{col}_{n+1}(P_\gamma^{-1})\Big(\sum_{i=1}^{\mu}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_n(t)}\varphi(x(t))\hat{\theta}(t)\\
&\quad + \sum_{i=1}^{\mu}\sum_{j=1}^{n}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_j(t)}x_{j+1}(t)\Big)\\
&\quad + s'(t)\mathrm{col}_{n+1}(P_\gamma^{-1})\varphi(x(t))\varphi'(x(t))\,\mathrm{col}_n^{'}(P_\gamma^{-1})s(t)\\
&\quad + s'(t)\mathrm{col}_{n+1}(P_\gamma^{-1})\varphi(x(t))\varphi'(x(t))\\
&\quad \times\sum_{i=1}^{\mu}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_n(t)}\mathrm{col}_{n+1}^{'}(P_\gamma^{-1})s(t)
\end{aligned}
\tag{17}
$$

where P_γ is the solution of the Lyapunov matrix equation [16]

$$
-\gamma P_\gamma - P_\gamma A'_{n+1} - A_{n+1}P_\gamma + B_{n+1,n+1}B'_{n+1,n+1} = 0,
\tag{18}
$$

for some $\gamma > 0$ and $s(t)$ is a column vector defined as

$$
\begin{aligned}
s_i(t) &= x_i(t) - r^{(i-1)}(t);\ 1\le i\le n,\\
s_{n+1}(t) &= x_{n+1} + \varphi(x(t))\hat{\theta}(t) - r^{(n)}(t).
\end{aligned}
\tag{19}
$$

Then $\lim_{t\to\infty} x_i(t) - r^{(i-1)}(t) = 0$, $1\le i\le n$.

Proof: Define $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$, $\tilde{p}(t) = p - \hat{p}(t)$, and $\tilde{\rho}(t) = \frac{1}{p_4} - \hat{\rho}(t)$. Then, we have

$$
\begin{aligned}
\dot{s}(t) &= A_{n+1}\,s(t)\\
&\quad + \Big(B_{n+1,n}\varphi(x(t)) + B_{n+1,n+1}G(x(t),\hat{\theta}(t))\Big)\tilde{\theta}(t)\\
&\quad + B_{n+1,n+1}\Big(\psi(x_{n+1}(t),u(t),\dot{u}(t))p + \varphi(x(t))\dot{\hat{\theta}}(t)\\
&\quad + g(x(t),\hat{\theta}(t)) - r^{(n+1)}(t)\Big)\\
&\quad + p_4 B_{n+1,n+1}\frac{\ddot{u}(t)}{(1+u^2(t))^2}.
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
g(x(t),\hat{\theta}) &= \sum_{i=1}^{\mu}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_n(t)}\varphi(x(t))\hat{\theta}(t)\\
&\quad + \sum_{i=1}^{\mu}\sum_{j=1}^{n}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_j(t)}x_{j+1}(t),\\
G(x(t),\hat{\theta}(t)) &= \sum_{i=1}^{\mu}\hat{\theta}_i\frac{\partial\varphi_i(x(t))}{\partial x_n(t)}\varphi(x(t)).
\end{aligned}
\tag{21}
$$

Under the action of the adaptive controller (16), which is a direct realization of the feedback

$$
\begin{aligned}
\ddot{u}(t) &= -\hat{\rho}(t)(1+u^2(t))^2\Big(\mathrm{col}_{n+1}^{'}(P_\gamma^{-1})\,s(t)\\
&\quad + \psi(x_{n+1}(t),u(t),\dot{u}(t))\hat{p}\\
&\quad + \sum_{i=1}^{\mu}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_n(t)}\varphi(x(t))\hat{\theta}(t)\\
&\quad + \sum_{i=1}^{\mu}\sum_{j=1}^{n}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_j(t)}x_{j+1}(t)\\
&\quad - r^{(n+1)}(t) + \varphi(x(t))\varphi'(x(t))\,\mathrm{col}_n^{'}(P_\gamma^{-1})s(t)\\
&\quad + \varphi(x(t))\varphi'(x(t))\sum_{i=1}^{\mu}\hat{\theta}_i(t)\frac{\partial\varphi_i(x(t))}{\partial x_n(t)}\mathrm{col}_{n+1}^{'}(P_\gamma^{-1})s(t)\Big).
\end{aligned}
\tag{22}
$$

Then, the dynamics of system (20) becomes

$$
\begin{aligned}
\dot{s}(t) &= \Big(A_{n+1} - B_{n+1,n+1}B'_{n+1,n+1}P_\gamma^{-1}\Big)s(t)\\
&\quad + \Big(B_{n+1,n}\varphi(x(t)) + B_{n+1,n+1}G(x(t),\theta(t))\Big)\tilde{\theta}(t)\\
&\quad + B_{n+1,n+1}\psi(x_{n+1}(t),u(t),\dot{u}(t))\,\tilde{p}(t)\\
&\quad + p_4\tilde{\rho}(t)B_{n+1,n+1}B'_{n+1,n+1}P_\gamma^{-1}s(t)\\
&\quad - p_4\tilde{\rho}(t)B_{n+1,n+1}r^{(n+1)}(t)\\
&\quad + p_4\tilde{\rho}(t)B_{n+1,n+1}\psi(x_{n+1}(t),u(t),\dot{u}(t))\hat{p}(t)\\
&\quad + p_4\tilde{\rho}(t)B_{n+1,n+1}g(x(t),\hat{\theta}(t))\\
&\quad + p_4\tilde{\rho}(t)\varphi(x(t))\dot{\hat{\theta}}(t).
\end{aligned}
\tag{23}
$$

Let us associate to the dynamics (23), the Lyapunov function candidate

$$
V = s'(t)P_\gamma^{-1}s(t) + \tilde{\theta}'(t)\tilde{\theta}(t) + p_4\tilde{\rho}^2(t) + \tilde{p}'(t)\tilde{p}(t).
\tag{24}
$$

Then the time-derivative of V along the trajectories of system (23) is

$$
\begin{aligned}
\dot{V} &= s'(t)\Big(A'_{n+1}P_\gamma^{-1} + P_\gamma^{-1}A_{n+1}\\
&\quad - 2P_\gamma^{-1}B_{n+1,n+1}B'_{n+1,n+1}P_\gamma^{-1}\Big)s(t)\\
&\quad + 2s'(t)P_\gamma^{-1}B_{n+1,n+1}\psi(x_{n+1}(t),u(t),\dot{u}(t))\,\tilde{p}(t)\\
&\quad + 2s'(t)P_\gamma^{-1}\Big(B_{n+1,n}\varphi(x(t))\\
&\quad + B_{n+1,n+1}G(x(t),\hat{\theta}(t))\Big)\tilde{\theta}(t)\\
&\quad + 2\tilde{\rho}(t)p_4 s'(t)P_\gamma^{-1}B_{n+1,n+1}B'_{n+1,n+1}P_\gamma^{-1}s(t)\\
&\quad - 2p_4\tilde{\rho}(t)s'(t)P_\gamma^{-1}B_{n+1,n+1}r^{(n+1)}(t)\\
&\quad + 2p_4\tilde{\rho}(t)s'(t)P_\gamma^{-1}B_{n+1,n+1}\psi(x_{n+1}(t),u(t),\dot{u}(t))\hat{p}(t)\\
&\quad + 2p_4\tilde{\rho}(t)s'(t)P_\gamma^{-1}B_{n+1,n+1}g(x(t),\hat{\theta}(t))\\
&\quad + 2p_4\tilde{\rho}(t)s'(t)P_\gamma^{-1}B_{n+1,n+1}\varphi(x(t))\dot{\hat{\theta}}(t)\\
&\quad - 2\tilde{\theta}'(t)\dot{\hat{\theta}}(t) - 2p_4\tilde{\rho}(t)\dot{\hat{\rho}}(t) - 2\tilde{p}'(t)\dot{\hat{p}}(t).
\end{aligned}
\tag{25}
$$

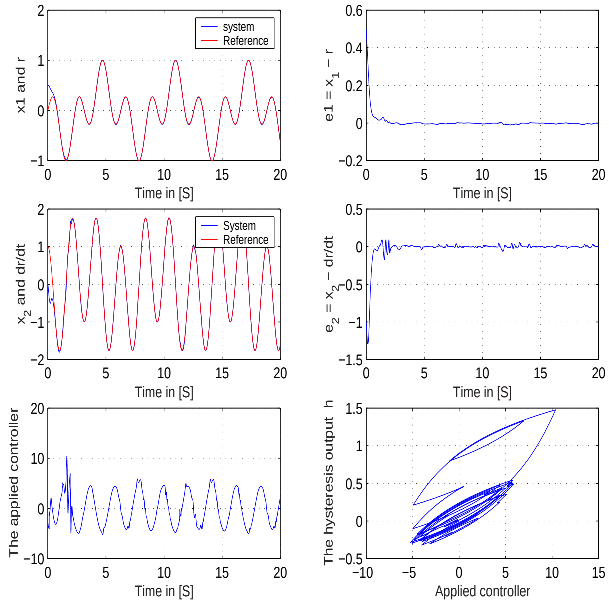

Fig. 3. The performance of the tracking error

Since the product $P_\gamma^{-1} B_{n+1,j}$ extracts the j column of P_γ^{-1}, and using Eqs. (16) and (18) then, we have

$$\dot{V} = s'(t)(-\gamma P_\gamma^{-1} - P_\gamma^{-1} B_{n+1,n+1} B'_{n+1,n+1} P_\gamma^{-1}) s(t) \leq 0. \tag{26}$$

From the Lyapunov matrix equation (18), we have $-P_\gamma\Big(\frac{\gamma}{2} I_{n+1} + A_{n+1}\Big)' - \Big(\frac{\gamma}{2} I_{n+1} + A_{n+1}\Big) P_\gamma + B_{n+1,n+1} B'_{n+1,n+1} = 0$. Since the matrix $-\frac{\gamma}{2} I_{n+1} - A_{n+1}$ is Hurwitz for all $\gamma > 0$, then P_γ solution of (18) is positive definite matrix. Thus, from (26) and by the use of LaSalle theorem, we conclude that $\lim_{t\to\infty} s_i(t) = 0,\ 1 \leq i \leq n$. This ends the proof.

VII. Illustrative Example

In this section, we illustrate the design of adaptive compensation on a nonlinear uncertain system subject to hysteresis input of form (3) with an external perturbation term. The dynamics of the system is given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \theta_1 \frac{1 - e^{-x_1}}{1 + e^{-x_1}} - \theta_2 (x_2^2 + 2x_1) \sin x_2 - \frac{1}{2} \theta_3 x_1 \sin 3t \\ &\quad + h(t), \end{aligned} \tag{27}$$

where the term $\frac{1}{2}\theta_3 x_1 \sin 3t$; $\theta_3 = -1$ is assumed to be the system external perturbation that will not be taken into account while the design of the adaptation algorithm. The parameters $\theta_1 = 0.1$ and $\theta_2 = 0.1$ are considered as the system unknown parameters. For $h = 0$, the aforementioned system is unstable. The linearization around the origin is also unstable dynamics. The simulations presented in Fig. 3 is performed for the initial conditions $x_1(0) = 0.5$, $x_2(0) = 0$, $\hat{p} = 0$, $\hat{\theta} = 0$, $\hat{\rho} = 0$. The adaptive law (16) is conceived for $\gamma = 7$ to force the system output to follow the reference trajectory $r(t) = \sin(t)\cos(2t)$. The hysteresis parameters are $p_1 = 2$, $p_2 = 5.7$, $p_3 = 16$, $p_4 = 0.05$.

According to this simulation, we conclude that the trajectory tracking controller is able not only to handle the effect of the hysteresis nonlinearity, but also to diminish the effects of the external perturbation that has not been taken into account by the applied controller.

VIII. Conclusion

The main contribution of this paper is the development of a new dynamic model describing various types of hysteresis phenomenon. It is showed that the developed model is of a crucial help to solve tracking problems for a class of non-smooth dynamical systems. Thanks to the structure of the developed hysteresis model that involves the second derivative of the control input, an adaptive smooth controller turns out to be sufficient for adaptive output tracking. It is worthwhile to mention that the use of the model is not limited to control exercises, but also suitable for identification purposes as well.

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