Robust State Estimation of Linear Neutral-Type Delay Systems: A Convex Optimization Setting

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***Abstract*— This paper is concerned with the problem of robust observer design for linear systems with neutral-type time-delays. New sufficient delay-independent conditions are given to solve the observation issue under noisy output measurements. Stated as linear matrix inequalities conditions, these sufficient conditions enable the determination of the observer gains that guarantee both asymptotic convergence of the observer in case of noiseless measurements and robust filtering in case of presence of measurements errors. The proposed linear matrix inequality conditions are derived without any major approximation or assumption on the neutral type time-delay system which make the observer design straightforward and less conservative.**

***Index Terms*— Keywords: Neutral-type delay systems; Observers; Optimal filtering; Linear Matrix Inequalities (LMIs).**

I. INTRODUCTION

The stability and the stabilizability of neutral-type delay systems have received a revival of interest during the last decade, see for example [1], [2], [3], [4] and the references therein. Such systems appear in many practical engineering domains as distributed networks containing lossless transmission lines, chemical engineering reactor applications, ship stabilization and VLSI systems [5], [6], [7]. Even though considerable research efforts have been undertaken on various aspects of dynamical systems with time delays [8], [9], the observation issue of systems with neutral-type delays has received a little attention. The available results on filtering and observation of neural time delay systems can be broadly classified into delay dependent and delay independent techniques, see for instance [10]. Despite the fact that delay dependent observer design is considered less conservative, the delay independent techniques remain preferable for their robustness and highly satisfactory performances.

In this paper the problem of observer design for neutral-type delay systems is addressed. In case of noiseless measurements, the proposed observer is merely a Luenberger observer having a classical proportional output injection term. For this case, we give sufficient linear matrix inequality condition that guarantees the existence of the observer gain. Subsequently, the problem of robustness against measurement errors is tackled. To deal with noisy measurements, the Luenberger observer is transformed into an integral observer that uses the integral path of the system and the observer outputs. This observer does not use the proportional output injection term as classical proportional integral observers do. For this reason, noise cannot be amplified even for high values of observer gains. In our design, although the delay is assumed to be known, the computation of the observer gain is totally independent from the amount of the system delay.

Throughout this paper, $\|\cdot\|$ stands for the usual Euclidean norm. The notation $A>0$ (respectively $A<0$) means that the matrix A is positive definite (respectively negative definite). We denote by $A^\top$ the matrix transpose of A. We note by I and $\mathbf{0}$ the identity matrix, and the null matrix of appropriate dimensions, respectively. "$\star$" is used to notify an element which is induced by transposition.

II. OBSERVER DESIGN

Consider the neutral-type delay system

$$\begin{array}{l} \dot{x}(t) - D\,\dot{x}(t-h) = A\,x(t) + A_d\,x(t-h) + Bu(t), \\ y(t) = C\,x(t) + D_1\xi(t), \end{array} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the input vector, and $y(t)\in\mathbb{R}^p$ is the system output. $D\in\mathbb{R}^{n\times n}$, $A\in\mathbb{R}^{n\times n}$, $A_d\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$, and $D_1\in\mathbb{R}^{p\times p}$ are constant matrices and h is a constant delay that appears in both the derivative and the delay state matrices. We assume that $\|D\|\le 1$ and $\xi(t)\in\mathbb{R}^p$ is a norm-bounded uncertainty that describes usually the output measurement errors. We assume that $(A,\,C)$ is an observable pair. The initial condition of system (1) is given by

$$x(t_0+\theta) = \varphi(\theta), \;\; \forall\theta\in[-h,\;0]\,. \tag{2}$$

The objective is to design an asymptotic observer for system (1) by setting $\xi(t)\equiv 0$. For this purpose, we set the dynamics of the observer as

$$\begin{aligned} \dot{\hat{x}}(t) - D\,\dot{\hat{x}}(t-h) &= A\,\hat{x}(t) + A_d\,\hat{x}(t-h) + Bu(t) \\ &\quad + P^{-1}Y\,(C\hat{x}(t)-y(t))\,, \end{aligned} \tag{3}$$

where $P\in\mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix and $Y\in\mathbb{R}^{n\times p}$ is a real arbitrary matrix to be determined

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later. Let $e(t)=\hat{x}(t)-x(t)$ be the observation error. Then we have

$$\dot{e}(t)-D\dot{e}(t-h)=\big(A+P^{-1}YC\big)\,e(t)+A_d\,e(t-h). \quad (4)$$

Consider the Lyapunov-Krasovskii functional candidate

$$\begin{aligned} V(e(t)) &= (e(t)-De(t-h))^\top P(e(t)-De(t-h)) \\ &\quad + \int_{t-h}^{t} e^\top(\tau)Q,e(\tau)\,\mathrm{d}\,\tau, \end{aligned} \quad (5)$$

where $P\in\mathbb{R}^{n\times n}$ and $Q\in\mathbb{R}^{n\times n}$ are symmetric and positive definite matrices. Then the time derivative of $V(e(t))$ along the trajectories of (4) is given by

$$\begin{aligned} & e^\top(t)\left(A^\top P+PA+YC+C^\top Y^\top+Q\right)e(t) \\ & + e^\top(t-h)\left(-Q-D^\top PA_d-A_d^\top PD\right)e(t-h) \\ & - e^\top(t)\left(C^\top Y^\top+A^\top P\right)De(t-h) \\ & - e^\top(t-h)D^\top\left(YC+PA\right)e(t) \\ & + e^\top(t-h)A_d^\top Pe(t)+e^\top(t)PA_de(t-h). \end{aligned} \quad (6)$$

If we suppose that $\frac{1}{2}Q+D^\top PA_d+A_d^\top PD>0$, then $\dot{V}(e(t))$ can be rewritten as

$$\begin{aligned} \dot{V} &= e^\top(t)\Big[A^\top P+PA+YC+C^\top Y^\top+Q \\ &\quad + PA_d\left(\frac{1}{2}Q+D^\top PA_d+A_d^\top PD\right)^{-1}A_d^\top P\Big]e(t) \\ &\quad - \left[\left(\frac{1}{2}Q+D^\top PA_d+A_d^\top PD\right)e(t-h)-A_d^\top Pe(t)\right]^\top \\ &\quad \times\left(\frac{1}{2}Q+D^\top PA_d+A_d^\top PD\right)^{-1} \\ &\quad \times\left[\left(\frac{1}{2}Q+D^\top PA_d+A_d^\top PD\right)e(t-h)-A_d^\top Pe(t)\right] \\ &\quad - e^\top(t)\left(C^\top Y^\top+A^\top P\right)De(t-h) \\ &\quad - e^\top(t-h)D'\left(YC+PA\right)e(t)-e^\top(t-h)\frac{Q}{2}e(t-h). \end{aligned} \quad (7)$$

This gives

$$\begin{aligned} \dot{V} &\le e^\top(t)\Big[A^\top P+PA+YC+C^\top Y^\top+Q \\ &\quad + PA_d\left(\frac{1}{2}Q+D^\top PA_d+A_d^\top PD\right)^{-1}A_d^\top P\Big]e(t) \\ &\quad - e^\top(t)\left(C^\top Y^\top+A^\top P\right)De(t-h) \\ &\quad - e^\top(t-h)D^\top\left(YC+PA\right)e(t)-e^\top(t-h)\frac{Q}{2}e(t-h) \\ &= \begin{bmatrix} e(t) \\ e(t-h)\end{bmatrix}^\top\times \\ &\quad \begin{bmatrix} \mathscr{L}_{1,1}(P,Y,Q) & -\left(C^\top Y^\top+A^\top P\right)D \\ -D^\top\left(YC+PA\right) & -\dfrac{Q}{2}\end{bmatrix} \\ &\quad \times\begin{bmatrix} e(t) \\ e(t-h)\end{bmatrix}, \end{aligned} \quad (8)$$

where $\mathscr{L}_{1,1}(P,Y,Q)=A^\top P+PA+YC+C^\top Y^\top+Q$ $+PA_d\big(\frac{1}{2}Q+D^\top PA_d+A_d^\top PD\big)^{-1}A_d^\top P$. Then $\dot{V}(e(t))<0$ if

$$\begin{bmatrix} \mathscr{L}_{1,1}(P,Y,Q) & -\left(C^\top Y^\top+A^\top P\right)D \\ -D^\top\left(YC+PA\right) & -\dfrac{Q}{2}\end{bmatrix}<0. \quad (9)$$

By the Schur complement lemma, the block $\mathscr{L}_{1,1}(P,Y,Q)<0$ if

$$\begin{bmatrix} A^\top P+PA+YC+C^\top Y^\top+Q & PA_d \\ A_d^\top P & -\frac{Q}{2}-D^\top PA_d-A_d^\top PD\end{bmatrix}<0, \quad (10)$$

Then we can write that $\dot{V}(e(t))<0$ if the following linear matrix inequality holds

$$\begin{bmatrix} \mathscr{W}_{1,1}(P,Y,Q) & PA_d & -(C^\top Y^\top+A^\top P)D \\ \star & -\dfrac{Q}{2}-D^\top PA_d-A_d^\top PD & \mathbf{0} \\ \star & \star & -\dfrac{Q}{2}\end{bmatrix}<0, \quad (11)$$

where $\mathscr{W}_{1,1}(P,Y,Q)=A^\top P+PA+YC+C^\top Y^\top+Q$.

Theorem 1: Consider system (1) with $\xi(t)\equiv 0$. Then if there exist two symmetric and positive definite matrices $P\in\mathbb{R}^{n\times n}$, $Q\in\mathbb{R}^{n\times n}$ and a matrix $Y\in\mathbb{R}^{n\times p}$ such that the linear matrix inequality (11) holds. Then the states of system (3) converge asymptotically to the states of system (1) when time elapses.

Theorem 1 gives a constructive method for designing the observer gain $K=P^{-1}Y$ via the solution of the linear matrix inequality (11) which is numerically tractable by any commercial software. Furthermore, the amount of delay does not appear in the LMI (11) which makes the observer valid for different values of the time-delay h. However, the knowledge of h remains necessary to build the dynamics of the asymptotic observer. Even though the amount of delay is not explicitly present in (11), the delay may affect considerably the performance of the observer. Remark that the condition $\frac{Q}{2}+D^\top PA_d+A_d^\top PD>0$ that we have imposed in the previous development appears in the diagonal of the matrix of inequality (11). For this reason, it is sufficient to fulfil condition (11) to obtain the observer gain. It is important to outline that the linear matrix inequality (11) is not conservative since it is not issued from any approximation of the terms that appear in (7).

III. ROBUST INTEGRAL OBSERVER DESIGN

Usually, the design of high-gain observers leads to noise amplification, and hence, the estimates cannot be filtered without a complete redesign of the observer gain. To clarify this fact, let us consider system (1) subject to the output uncertainty $\xi(t)$. Then if we use observer (3), the dynamics of the observation error is given by

$$\begin{aligned}\dot{e}(t) - D\dot{e}(t-h) = \left(A + P^{-1}YC\right) e(t) + A_d e(t-h) \\ - P^{-1}YD_1\xi(t).\end{aligned} \tag{12}$$

It is clear that if the stability of the observation error given by (12) requires a high-gain vector $K = P^{-1}Y$, then the value of the uncertainty in (12) shall be amplified. For this reason, the trade off between stability and filtering remains unsolvable. Our aim is to decouple the effect of noise from the observer gain. For this purpose, we shall feed back the observer with the first integral of the system and the observer outputs. The notion of this kind of observers has been introduced in [11] for both single output linear and nonlinear systems. The reader is also referred to the references [12], [13], [14] to see other types of proportional and integral observers. Let us consider the augmented system

$$\begin{aligned}\dot{x}(t) - D\,\dot{x}(t-h) &= A\,x(t) + A_d\,x(t-h) + Bu(t),\\ \dot{\eta}(t) &= C\,x(t) + D_1\xi(t),\end{aligned} \tag{13}$$

where $x(t_0+\theta) = \varphi(\theta)$, $\forall\theta \in [-h, \quad 0]$, $\eta(t_0) = 0$, and $\eta(t) = \displaystyle\int_{t_0}^{t} Cx(s) + D_1\xi(s)\ \mathrm{d}\,s$ is the new output of the neutral delay system. Let $z(t) = \left[\begin{array}{cc}\eta(t) & x(t)\end{array}\right]^\top$ be the new state vector and define

$$\begin{aligned}\widetilde{A} &= \left[\begin{array}{cc}\mathbf{0} & C\\ \mathbf{0} & A\end{array}\right],\ \widetilde{D} = \left[\begin{array}{cc}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & D\end{array}\right],\\ \widetilde{D}_1 &= \left[\begin{array}{c}D_1\\ \mathbf{0}\end{array}\right],\ \widetilde{C}' = \left[\begin{array}{c}I\\ \mathbf{0}\end{array}\right],\\ \widetilde{B} &= \left[\begin{array}{c}\mathbf{0}\\ B\end{array}\right],\ \widetilde{A}_d = \left[\begin{array}{cc}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & A_d\end{array}\right],\end{aligned} \tag{14}$$

as the new system matrices of dimensions $(n+p)\times(n+p)$, $(n+p)\times(n+p)$, $(n+p)\times p$, $(n+p)\times p$, $(n+p)\times m$, $(n+p)\times(n+p)$, respectively. Consider $\eta(t)$ as the new output vector of system (13). Then, we write

$$\begin{aligned}\dot{z}(t) - \widetilde{D}\dot{z}(t-h) &= \widetilde{A}z(t) + \widetilde{A}_d z(t-h) + \widetilde{B}u(t)\\ &\quad + \widetilde{D}_1\xi(t),\\ \widetilde{y}(t) &= \widetilde{C}z(t).\end{aligned} \tag{15}$$

By taking the integral of the output as the new output, we translate the uncertainty $\xi(t)$ to the state dynamics, see (15). Hence, any high-gain observer for system (15) will act as a filter because noise is viewed now as a system uncertainty.

The dynamics of the observer is readily constructed as

$$\begin{aligned}\dot{\hat{z}}(t) - \widetilde{D}\dot{\hat{z}}(t-h) &= \widetilde{A}\hat{z}(t) + \widetilde{A}_d\hat{z}(t-h) + \widetilde{B}u(t)\\ &\quad + \widetilde{P}^{-1}\widetilde{Y}\left(\widetilde{C}\hat{z}(t) - \widetilde{y}(t)\right),\end{aligned} \tag{16}$$

and hence, the dynamics of the observation error $e(t) = \hat{z}(t) - z(t)$ becomes

$$\dot{e}(t) - \widetilde{D}\dot{e}(t-h) = \left(\widetilde{A} + \widetilde{P}^{-1}\widetilde{Y}\widetilde{C}\right) e(t) + \widetilde{A}_d e(t-h) - \widetilde{D}_1\xi(t). \tag{17}$$

Even though the new observer dynamics (16) is in form (4), the injection term of (16) is an integral path of the difference of the observer and the system outputs. It remains now to deal with the calculation of the observer gain so as to ensure the asymptotic stability of the observation error when $\xi(t) \equiv 0$ and satisfy the following performance index for all initial conditions $e(s),\ -h \le s \le 0$ and $\forall t \ge 0$

$$\int_0^t \left\{ e^\top(s)\widetilde{C}'\widetilde{C}e(s) - \gamma^2\xi^\top(s)\xi(s)\right\}\ \mathrm{d}\,s \le \mathscr{V}(0); \tag{18}$$

where $\mathscr{V}(0) = (e(0) - \widetilde{D}e(-h))^\top\widetilde{P}(e(0) - \widetilde{D}e(-h)) + \int_{-h}^{0} e^\top(\tau)\widetilde{Q}e(\tau)\mathrm{d}\,\tau$, and $\widetilde{P}$, $\widetilde{Q}$ are symmetric and positive definite matrices of appropriate dimensions. It is obvious that if the initial conditions $e(t) = 0$ for $-h \le t \le 0$, then the performance index (18)is equivalent to $\|\widetilde{C}e(t)\| \le \gamma\|\xi(t)\|$. Setting the performance index in form (18) is realistic since the initial condition of the system is generally unknown in such observation problems. We summarize the result of this section in the following statement.

Theorem 2: The observer error dynamics (17) is asymptotically stable for $\xi(t) \equiv 0$ and verifies condition (18) for $\xi(t) \not\equiv 0$ if there exist two positive and definite matrices $\widetilde{P} \in \mathbb{R}^{(n+p)\times(n+p)}$, $\widetilde{Q} \in \mathbb{R}^{(n+p)\times(n+p)}$, a matrix $\widetilde{Y} \in \mathbb{R}^{(n+p)\times p}$, and a positive constant γ^2 such that the following LMI holds

$$\left[\begin{array}{cccc}\mathscr{M}_{1,1} & \widetilde{P}\widetilde{A}_d & -(\widetilde{C}^\top\widetilde{Y}^\top + \widetilde{A}^\top\widetilde{P})\widetilde{D} & -\widetilde{P}\widetilde{D}_1\\ \star & -\frac{\widetilde{Q}}{2} - \widetilde{D}^\top\widetilde{P}\widetilde{A}_d - \widetilde{A}_d^\top\widetilde{P}\widetilde{D} & \mathbf{0} & \mathbf{0}\\ \star & \star & -\frac{\widetilde{Q}}{2} & \widetilde{D}^\top\widetilde{P}\widetilde{D}_1\\ \star & \star & \star & -\gamma^2 I\end{array}\right] < 0, \tag{19}$$

where $\mathscr{M}_{1,1} = \widetilde{A}^\top\widetilde{P} + \widetilde{P}\widetilde{A} + \widetilde{Y}\widetilde{C} + \widetilde{C}^\top\widetilde{Y}^\top + \widetilde{C}^\top\widetilde{C} + \widetilde{Q}$.

Proof. Let $\mathscr{V}(e(t)) = (e(t) - \widetilde{D}e(t-h))^\top\widetilde{P}(e(t) - \widetilde{D}e(t-h)) + \displaystyle\int_{t-h}^{t} e^\top(\tau)\widetilde{Q}e(\tau)\ \mathrm{d}\,\tau$. Then, we have

$$\begin{aligned}&\int_0^t \left\{ e^\top(s)\widetilde{C}^\top\widetilde{C}e(s) - \gamma^2\xi^\top(s)\xi(s)\right\}\ \mathrm{d}\,s - \mathscr{V}(0)\\ &\le \int_0^t \left\{ e^\top(s)\widetilde{C}^\top\widetilde{C}e(s) - \gamma^2\xi^\top(s)\xi(s)\right\}\ \mathrm{d}\,s + \mathscr{V}(e(t))\\ &- \mathscr{V}(0)\\ &= \int_0^t \left\{ e^\top(s)\widetilde{C}^\top\widetilde{C}e(s) - \gamma^2\xi^\top(s)\xi(s) + \dot{\mathscr{V}}(e(s))\right\}\ \mathrm{d}\,s.\end{aligned}$$

Under the assumption that $\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top\widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top\widetilde{P}\widetilde{D} > 0$, we have

$$
\begin{aligned}
& e^\top(t)\widetilde{C}^\top\widetilde{C}e(t) - \gamma^2\xi^\top(t)\xi(t) + \dot{\mathscr{V}}(e(t)) \\
&= e^\top(t)\widetilde{C}^\top\widetilde{C}e(t) - \gamma^2\xi^\top(t)\xi(t) \\
&+ e^\top(t)\Big[A^\top\widetilde{P} + \widetilde{P}A + \widetilde{Y}\widetilde{C} + \widetilde{C}^\top\widetilde{Y}^\top + \widetilde{Q} + \widetilde{P}\widetilde{A}_d \\
&\times \Big(\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top\widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top\widetilde{P}\widetilde{D}\Big)^{-1}\widetilde{A}_d^\top\widetilde{P}\Big]e(t) \\
&- \Big[\Big(\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top\widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top\widetilde{P}\widetilde{D}\Big)e(t-h) - \widetilde{A}_d^\top\widetilde{P}e(t)\Big]^\top \\
&\times \Big(\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top\widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top\widetilde{P}\widetilde{D}\Big)^{-1} \times \\
&\Big[\Big(\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top\widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top\widetilde{P}\widetilde{D}\Big)e(t-h) - \widetilde{A}_d^\top\widetilde{P}e(t)\Big] \\
&- e^\top(t)\Big(\widetilde{C}^\top\widetilde{Y}^\top + \widetilde{A}^\top\widetilde{P}\Big)\widetilde{D}e(t-h) \\
&- e^\top(t-h)\widetilde{D}^\top\Big(\widetilde{Y}\widetilde{C} + \widetilde{P}\widetilde{A}\Big)e(t) \\
&- e^\top(t-h)\frac{\widetilde{Q}}{2}e(t-h) \\
&- \xi^\top(t)\widetilde{D}_1^\top\widetilde{P}e(t) + \xi^\top(t)\widetilde{D}_1^\top\widetilde{P}\widetilde{D}e(t-h) \\
&- e^\top(t)\widetilde{P}\widetilde{D}_1\xi(t) + e^\top(t-h)\widetilde{D}^\top\widetilde{P}\widetilde{D}_1\xi(t) \\
&\le \zeta^\top(t)\begin{bmatrix} \mathscr{L}_{1,1}(\widetilde{P},\widetilde{Y},\widetilde{Q}) + \widetilde{C}^\top\widetilde{C} & -(\widetilde{C}^\top\widetilde{Y}^\top + \widetilde{A}^\top\widetilde{P})\widetilde{D} & -\widetilde{P}\widetilde{D}_1 \\ -\widetilde{D}^\top(\widetilde{Y}\widetilde{C} + \widetilde{P}\widetilde{A}) & -\frac{\widetilde{Q}}{2} & \widetilde{D}^\top\widetilde{P}\widetilde{D}_1 \\ -\widetilde{D}_1^\top\widetilde{P} & \widetilde{D}_1^\top\widetilde{P}\widetilde{D} & -\gamma^2 I \end{bmatrix}\zeta(t),
\end{aligned} \tag{20}
$$

where $\zeta(t) = \begin{bmatrix} e(t) & e(t-h) & \xi(t) \end{bmatrix}^\top$, and $\mathscr{L}_{1,1}(\widetilde{P},\widetilde{Y},\widetilde{Q}) = A^\top\widetilde{P} + \widetilde{P}A + \widetilde{Y}\widetilde{C} + \widetilde{C}^\top\widetilde{Y}^\top + \widetilde{Q} + \widetilde{P}\widetilde{A}_d\Big(\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top\widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top\widetilde{P}\widetilde{D}\Big)^{-1}\widetilde{A}_d^\top\widetilde{P}$. Then the optimality condition (18) is satisfied if

$$
\begin{bmatrix} \mathscr{L}_{1,1}(\widetilde{P},\widetilde{Y},\widetilde{Q}) + \widetilde{C}^\top\widetilde{C} & -(\widetilde{C}^\top\widetilde{Y}^\top + \widetilde{A}^\top\widetilde{P})\widetilde{D} & -\widetilde{P}\widetilde{D}_1 \\ -\widetilde{D}^\top(\widetilde{Y}\widetilde{C} + \widetilde{P}\widetilde{A}) & -\frac{\widetilde{Q}}{2} & \widetilde{D}^\top\widetilde{P}\widetilde{D}_1 \\ -\widetilde{D}_1^\top\widetilde{P} & \widetilde{D}_1^\top\widetilde{P}\widetilde{D} & -\gamma^2 I \end{bmatrix} < 0. \tag{21}
$$

By applying the Schur complement result, condition (21) is equivalent to (19). It is always interesting to find the smallest value of γ that verifies inequality (19). In this case, the linear optimization problem (19) must be modified to $\min\limits_{\widetilde{P},\widetilde{Y},\widetilde{Q}} \ \gamma \ \text{ s.t. } (19)$.

IV. ILLUSTRATIVE EXAMPLE

The dynamics of a lossless transmission line is modelled by the following neutral-type delay system [6]

$$
\begin{aligned}
\dot{x}(t) - \begin{bmatrix} 0 & \alpha_4 & 0 & 0 \\ \alpha_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\dot{x}(t-h)
&= \begin{bmatrix} -\alpha_0 & 0 & \alpha_0 & 0 \\ 0 & -\alpha_1 & 0 & -\alpha_1 \\ -\alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_3 & 0 & 0 \end{bmatrix}x(t) \\
&+ \begin{bmatrix} 0 & \alpha_0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2\,\alpha_4 & 0 & 0 \\ -\alpha_3\,\alpha_5 & 0 & 0 & 0 \end{bmatrix}x(t-h) \\
&+ \begin{bmatrix} 0 & \beta_0 \\ 0 & 0 \\ \alpha_2\beta_0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\
y(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}x(t) \\
&+ \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\xi(t)
\end{aligned} \tag{22}
$$

where the system parameters are defined as $\alpha_0 = \sqrt{c}/(c_0 R_0\sqrt{c} + c_0\sqrt{L})$, $\alpha_1 = \sqrt{c}/(c_1 R_1\sqrt{c} + c_1\sqrt{L})$, $\alpha_2 = (R_0\sqrt{c} + \sqrt{L})/(L_0\sqrt{c})$, $\alpha_3 = (R_1\sqrt{c_1} + \sqrt{L})/(L_1\sqrt{c})$, $\alpha_4 = (R_0\sqrt{c} - \sqrt{L})/(R_0\sqrt{c} + \sqrt{L})$, $\alpha_5 = (R_1\sqrt{c} - \sqrt{L})/(R_1\sqrt{c} + \sqrt{L})$, $u_2(t) = \dot{u}_1(t)$, $h = \sqrt{c}\,L$. The numerical values of the system parameters are: $L_1 = 0.2$ $[H]$, $L = 1$ $[m]$, $h = 0.1414$ $[S]$, $c = 0.02$ $[S^2/m]$, $L_0 = 0.1$ $[H]$, $\beta_0 = 0.01$, $d_1 = 0.3$, $d_2 = 0.1$, $c_1 = 0.1$ $[F]$, $R_0 = 5$ $[Omhs]$, $R_1 = 10$ $[Omhs]$. To implement the robust time-delay observer (16) we shall delay the observer states by a constant delay h and consider the terms $\widetilde{A}_d\hat{z}(t-h)$ and $\widetilde{D}\dot{\hat{z}}(t-h)$ as feedback inputs to observer (16). This technique permits us to implement the observer dynamics as it appears without augmenting the order of the states. In Fig. 1 the noisy output of system (22) are reported when a periodic control input $u_1(t) = 5\,\sin(10\,t)$ [V] is applied to system (22). The initial values of system (22) are $(x_i)_{1\le i\le 4}(t) = -1$ for $t < h$. In Figs. 2 and 3, the behavior of the estimated states along with the real states of system (22) are represented. From these simulations, we see clearly that the observer states are quite insensitive to a band-limited white noise that comes corrupting the measurements. The maximum amplitude of noise is set to 10. The simulations are done after solving the filtering problem (19) with respect to P, Q, Y and γ. The solution is

$$P = \begin{bmatrix} 4.2567 & 0 & 0.4138 & 0 \\ 0 & 7.2728 & 0 & -0.8400 \\ 0.4138 & 0 & 2.5476 & 0 \\ 0 & -0.8400 & 0 & 2.7913 \\ -0.2236 & 0 & 0.0363 & 0 \\ 0 & -0.3903 & 0 & -0.0538 \end{bmatrix.$$

$$\left. \begin{matrix} -0.2236 & 0 \\ 0 & -0.3903 \\ 0.0363 & 0 \\ 0 & -0.0538 \\ 0.0195 & 0 \\ 0 & 0.0539 \end{matrix} \right],$$

$$Q = \left[\begin{matrix} 2.5839 & 0 & -0.1405 & 0 \\ 0 & 2.5769 & 0 & 0.1190 \\ -0.1405 & 0 & 6.0484 & 0 \\ 0 & 0.1190 & 0 & 6.2529 \\ -0.0070 & 0 & 0.1108 & 0 \\ 0 & -0.0049 & 0 & -0.0932 \end{matrix} \right.$$

$$\left. \begin{matrix} -0.0070 & 0 \\ 0 & -0.0049 \\ 0.1108 & 0 \\ 0 & -0.0932 \\ 0.2461 & 0 \\ 0 & 0.3951 \end{matrix} \right],$$

$$Y = \begin{bmatrix} -7.4319 & 0 \\ 0.0001 & -6.2441 \\ -18.2621 & 0.0002 \\ 0 & 18.4801 \\ -3.8303 & 0 \\ 0 & -6.8959 \end{bmatrix}, \ \gamma = 0.8803.$$

V. CONCLUSION

The problem of robust observer design for a class of systems with neutral-type time delays is addressed. It was shown that the proposed linear matrix inequalities conditions are not dependent upon certain class of linear neutral delay systems and enjoy the property to be less restrictive. Accordingly, extension of this work to neutral systems with multiple time-delays is also possible. We conjecture that dual results can be obtained in case of stabilization by static feedback and more optimality conditions can be imposed. This point will be the subject of future investigation.

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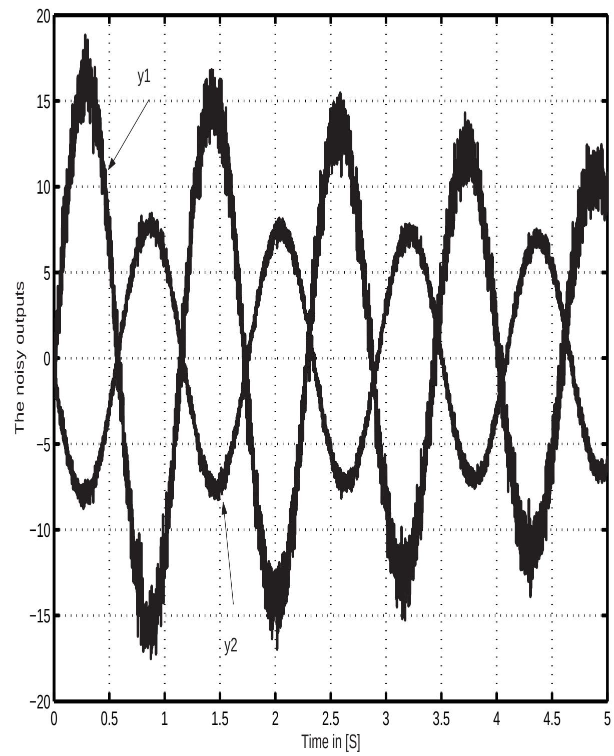

Fig. 1. The Noisy outputs

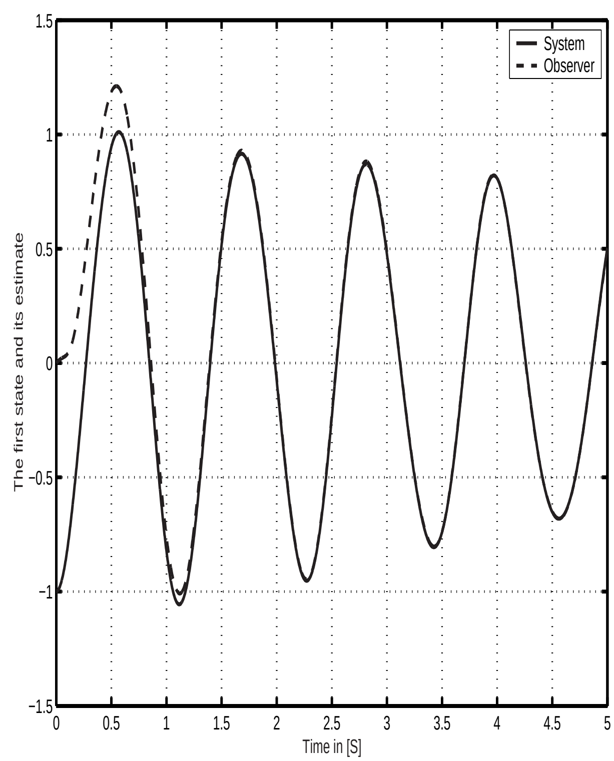

Fig. 2. The first state and its estimate

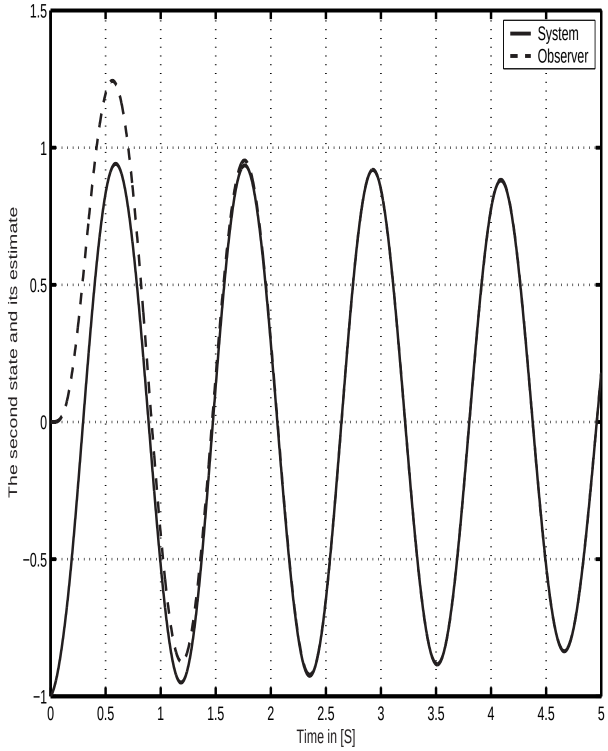

Fig. 3. The second state and its estimate

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