Robust Stabilization of Uncertain Aircraft Models*

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The premise underlying robust control is to explicitly model the mismatch between a nominal model and the true behavior of the physical system; this mismatch is accounted for by the plant "uncertainty". The largest advances in the theory of robust control have for the most part dealt with the question of analysis: given mathematical descriptions of the physical system, the system uncertainty, and the control system, determine whether the closed loop system performs to specifications for all possible values of the system uncertainty. If the answer to this question is positive, it is likely that the control system will perform to specifications when applied to the real system being controlled. In this note we propose a simple robust controller that stabilizes uncertain aerospace systems with arbitrary type of bounded uncertainties. The controller design takes into account the unmodeled input dynamics which may result from atmospheric perturbations. A new condition of quadratic stability is derived which makes easier the implementation of such robust controllers. An example of control of uncertain missile model is illustrated to highlight the robustness of the proposed method.

Key-Words. Linear uncertain systems; Robust control; Lypunov stability; Missile autopilot.

introduction

We are never able to guarantee in advance that any model-based algorithm will work in practice. Irrespective of how the models used for control are obtained, they will be imperfect. A way to reduce the risk somewhat is to construct reliable and robust controllers. The design is then based not only on nominal design models, but also on specified sets of deviations between the models and the true systems. Such sets are called error models, or uncertainty models.

The stabilization and control of uncertain plants is the first step for the construction of both autonomous and intelligent systems. The design of such a control system is often found challenging because of the insufficient knowledge and unmodeled dynamics of the system, external disturbance, and the inherent problem of sensor noise. Hence, the controller is required to be immune to such operating conditions. While these theoretical and computational tools for analysis have been a success by most measures, the corresponding quest for tools which tackle synthesis for uncertain systems has fallen short of expectations. The main reason for this lack of progress is that controller synthesis is a much harder, and less studied problem. Extensive research efforts include developing new theoretical and computational tools for the design of robust and optimal control systems such as H_∞ minimization theory, μ-synthesis theory, Linear matrix inequality theory and Lyapunov methods based techniques which are still in progress and have been found to be ideal for such applications.

Let us mention that Lyapunov based methods have a great relationship with optimal control theory which governs strategies for maximizing a performance measure or minimizing a cost function as the state of a dynamic system evolves. If the information that the control system must use is uncertain or if the dynamic system is forced by random disturbances, it may not be possible to optimize this criterion with certainty. The best one can hope to do is to maximize or minimize the expected value of the criterion, given assumptions about the statistics of the uncertain factors. This leads, of course, to the concept of stochastic optimal control that recognizes the random behavior of the system and that attempts to optimize response or stability on the average rather than with assured precision.

In this paper we provide a new sample control methodology for the stabilization of uncertain linear systems for which matching conditions are not satisfied. The unmodeled dynamics are described only in terms of bounds on their possible sizes. The proposed controller realizes the quadratic stability of the uncertain system with less computational tools and less restrictive conditions on the uncertain parts. The controller gain is issued from the solution of a parameter-dependent Lyapunov-like matrix equation and the tuning parameter permits to regulate the gain of the linear part of the controller in order to overcome the effect of uncertainties.

In the second section we formulate the problem with

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some preliminaries. Section 3 is mainly devoted to the main result of this paper and finally, an example of uncertain missile autopilot is shown to illustrate the control strategy. We note
$\mathbb{R}$ is the set of real numbers.
$\|\cdot\|$ denotes the habitual Euclidean norm.
A' is the matrix transpose of A.
$\lambda_{\min}(A)$ is the smallest eigenvalue of the matrix A.
$\lambda_{\max}(A)$ is the largest eigenvalue of the matrix A.
If A is a matrix, then $|A| = |a_{i,j}|, \quad 1 \leq i,\, j \leq n$.
$A > B$ i.e., the matrix $A - B$ is positive definite.
The measure of a matrix A is denoted by $\mu(A) = \lim_{\theta \to 0} \dfrac{(\|I + \theta A\| - 1)}{\theta}$. Depending upon the induced norm, several types of measures are given. We note

$$\mu_1(A) = \max_j \left(\mathrm{Re}(a_{i,i}) + \sum_{i,\, i \neq j} |a_{i,j}| \right),$$
$$\mu_2(A) = \max_i (\lambda_i(A + A')/2),$$
$$\mu_\infty(A) = \max_i \left(\mathrm{Re}(a_{i,i}) + \sum_{j,\, j \neq i} |a_{i,j}| \right).$$

Problem setup

The uncertain system is assumed to satisfy the following assumptions.

Assumption 1 *The system is expressed as*

$$\dot{x} = (A + E(t))\, x + B\,(u + \xi(t))\,, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of the state variables, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are the nominal matrices forming a controllable pair. $u \in \mathbb{R}^m$ is the control input and $\xi(t)$ stands for an external disturbance vector having the length of u.

Assumption 2 *The uncertainty matrix $E(t) \in \mathbb{R}^{n \times n}$ is supposed to be an arbitrary non measurable matrix but bounded as follows:*

$$E(t) \in \Omega \quad \text{for all } t \geq 0, \tag{2}$$

and

$$|E(t)| \leq \epsilon W, \tag{3}$$

where Ω is a compact set and $0 \in \Omega$. ϵ is a small positive parameter and $W \in \mathbb{R}^{n \times n}$ is a real matrix with positive parameters.

Assumption 3 *For all $t \geq 0$, the disturbance $\xi(t)$ is bounded as*

$$\|\xi(t)\| \leq \rho. \tag{4}$$

The problem is to steer the states of system (1) to the origin by designing a feedback controller u that defeats the effects of both the unmodeled dynamics and the external perturbation $\xi(t)$. The design of the stabilizing controller is achieved under the assumption that a full state measurement is possible. For general systems, this assumption is not valid, but in control of aerospace systems, the reliable and adequate sensors do not miss in such situations. The quadratic stability of uncertain systems subject to unmodeled dynamics is given by the following definition.[4]

Definition 1 *The system (1) is said to be quadratically stabilizable if there exists a continuous $v(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^m$, with $v(0) = 0$ an $n \times n$ positive definite matrix H and a constant $\beta > 0$ such that for any admissible uncertainty $E(t) \in \mathbb{F} \subset \mathbb{R}^{n \times n}$, for the Lyapunov function $V(x) = x'Hx$, the derivative $\dot{V}$, corresponding to the closed loop-system with the feedback law $u = v(x(t))$, satisfies the inequality*

$$\begin{aligned} \dot{V} = x' \left[(A + E(t))'\, H + H\,(A + E(t)) \right] x \\ +2x'H\,B\,v(\cdot) \leq -\beta \|x\|^2 \end{aligned} \tag{5}$$

for all pairs $(x,\, t)$.

Before giving the main result of this paper we would rather present the following theorems that we shall need later.

Theorem 1 *(See[2]) Let A and $B \in \mathbb{C}^{n \times n}$, and μ be defined as in the notation section. Then*

a) $\mu(I) = 1$, $\mu(-I) = -1$, $\mu(0) = 0$;

b) $-\mu(-A) \leq \mathrm{Re}\,(\lambda_i(A)) \leq \mu(A)$;

c) $\mu(c\,A) = c\,\mu(A)$, $\forall c \in \mathbb{R}$;

d) $\mu(A + B) \leq \mu(A) + \mu(B)$.

The proof of the next theorem is given in the reference.[1]

Theorem 2 *For any piecewise-continuous matrices $W(t)$ and $Y(t) \in \mathbb{R}^{n \times n}$, and $|W(t)| \leq Y(t)$, the matrix measures of the matrices $W(t)$, $|W(t)|$ and $Y(t)$ are well defined and have the property:*

$$\mu\,(W(t)) \leq \mu\,(|W(t)|) \leq \mu\,(Y(t))\,. \tag{6}$$

Main result

In this section we develop the nonlinear controller that stabilizes system (1) under the effects of unmodeled dynamics and external perturbations. The linear part of the controller is a high-gain linear controller that tries to overcome the unmodeled dynamics of system (1), while the nonlinear term is designed to kill the external perturbation. We shall give a necessary condition to chose the controller gain that permits to retain the quadratic stability of the uncertain system. We show that this condition is not restrictive compared with others stability conditions. The design of the stabilizing controller is given in the following theorem.

Theorem 3 *If the parameter γ is selected so as to the following conditions hold*

i) the matrix $-A' - \frac{\gamma}{2}I$ is Hurwitz and

ii) γ satisfies

$$\gamma + \lambda_{\min}\left(H^{-\frac{1}{2}}BB'H^{-\frac{1}{2}}\right) > \epsilon\mu\left(\left|H^{\frac{1}{2}}\right|W'\left|H^{-\frac{1}{2}}\right| + \left|H^{-\frac{1}{2}}\right|W\left|H^{\frac{1}{2}}\right|\right),$$

then system (1) satisfying assumptions 1-3 is quadratically stabilizable by the controller

$$u = -B'H^{-1}x - \frac{2\rho^2 B'H^{-1}x}{2\rho\left\|B'H^{-1}x\right\| + \epsilon^\star e^{-\beta t}},\ \epsilon^\star;\ \beta \in \mathbb{R}_+, \tag{7}$$

where H is the solution of the Lyapunov-like equation

$$-\gamma H - HA' - AH + BB' = 0. \tag{8}$$

Proof. Note that $H > 0$ always exists whenever the matrix $-A' - \frac{\gamma}{2}I$ is Hurwitz. This comes from the fact that equation (8) can be written as

$$\left(-A' - \frac{\gamma}{2}I\right)' H + H\left(-A' - \frac{\gamma}{2}I\right) = -BB', \tag{9}$$

which translates the Lyapunov stability of the matrix $-A' - \frac{\gamma}{2}I$. Let $V = x'H^{-1}x$ be the Lyapunov function candidate for the closed loop system

$$\begin{aligned}\dot{x} &= \left(A - BB'H^{-1}x + E(t)\right)x \\ &\quad -2\rho^2\frac{BB'H^{-1}x}{2\rho\left\|B'H^{-1}x\right\| + \epsilon^\star e^{-\beta t}} + B\xi(t),\end{aligned} \tag{10}$$

then

$$\begin{aligned}\dot{V} &= \dot{x}'H^{-1}x + x'H^{-1}\dot{x} \\ &= x'\left(A' - H^{-1}BB' + E'(t)\right)H^{-1}x \\ &\quad + x'H^{-1}\left(A - BB'H^{-1} + E(t)\right)x \\ &\quad - 4x'\frac{\rho^2H^{-1}BB'H^{-1}x}{2\rho\left\|B'H^{-1}x\right\| + \epsilon^\star e^{-\beta t}} + 2x'H^{-1}B\xi(t) \\ &= x'\left(A'H^{-1} + H^{-1}A - 2H^{-1}BB'H^{-1}\right. \\ &\quad \left.+E'(t)H^{-1} + H^{-1}E(t)\right)x \\ &\quad - 4x'\frac{\rho^2H^{-1}BB'H^{-1}x}{2\rho\left\|B'H^{-1}x\right\| + \epsilon^\star e^{-\beta t}} + 2x'H^{-1}B\xi(t).\end{aligned} \tag{11}$$

Multiplying both sides of equation (8) by H^{-1}, then the matrix H^{-1} verifies the following matrix equation

$$-\gamma H^{-1} - A'H^{-1} - H^{-1}A + H^{-1}BB'H^{-1} = 0. \tag{12}$$

substituting (12) in the last equation of (11), we have

$$\begin{aligned}\dot{V} &= x'\left(-\gamma H^{-1} - H^{-1}BB'H^{-1} + \right. \\ &\quad \left.+E'(t)H^{-1} + H^{-1}E(t)\right)x \\ &\quad - 4x'\frac{\rho^2H^{-1}BB'H^{-1}x}{2\rho\left\|B'H^{-1}x\right\| + \epsilon^\star e^{-\beta t}} + 2x'H^{-1}B\xi(t).\end{aligned} \tag{13}$$

Define $\|x\|^2_{H^{-1}} = x'H^{-1}x$, this gives

$$\begin{aligned}\dot{V} &\leq -\gamma\|x\|^2_{H^{-1}} - \lambda_{\min}\left(H^{-\frac{1}{2}}BB'H^{-\frac{1}{2}}\right)\|x\|^2_{H^{-1}} \\ &\quad + \lambda_{\max}\left(H^{\frac{1}{2}}\left(E'(t)H^{-1} + H^{-1}E(t)\right)H^{\frac{1}{2}}\right)\|x\|^2_{H^{-1}} \\ &\quad - 4x'\frac{\rho^2H^{-1}BB'H^{-1}x}{2\rho\left\|B'H^{-1}x\right\| + \epsilon^\star e^{-\beta t}} + 2x'H^{-1}B\xi(t).\end{aligned} \tag{14}$$

Consequently,

$$\begin{aligned}\dot{V} &\leq -\gamma\|x\|^2_{H^{-1}} - \lambda_{\min}\left(H^{-\frac{1}{2}}BB'H^{-\frac{1}{2}}\right)\|x\|^2_{H^{-1}} \\ &\quad + \lambda_{\max}\left(H^{\frac{1}{2}}E'(t)H^{-\frac{1}{2}} + H^{-\frac{1}{2}}E(t)H^{\frac{1}{2}}\right)\|x\|^2_{H^{-1}} \\ &\quad - 4x'\frac{\rho^2H^{-1}BB'H^{-1}x}{2\rho\left\|B'H^{-1}x\right\| + \epsilon^\star e^{-\beta t}} + 2\left\|x'H^{-1}B\right\|\left\|\xi(t)\right\|.\end{aligned} \tag{15}$$

Using assumption 3, we write

$$\begin{aligned}\dot{V} &\leq -\gamma\|x\|^2_{H^{-1}} - \lambda_{\min}\left(H^{-\frac{1}{2}}BB'H^{-\frac{1}{2}}\right)\|x\|^2_{H^{-1}} \\ &\quad + \lambda_{\max}\left(H^{\frac{1}{2}}E'(t)H^{-\frac{1}{2}} + H^{-\frac{1}{2}}E(t)H^{\frac{1}{2}}\right)\|x\|^2_{H^{-1}} \\ &\quad + 2\rho\left\|x'H^{-1}B\right\| - \frac{4\rho^2\left\|x'H^{-1}B\right\|^2}{2\rho\left\|B'H^{-1}x\right\| + \epsilon^\star e^{-\beta t}}\end{aligned} \tag{16}$$

Finally,

$$\begin{aligned}\dot{V} &\leq -\gamma\|x\|^2_{H^{-1}} - \lambda_{\min}\left(H^{-\frac{1}{2}}BB'H^{-\frac{1}{2}}\right)\|x\|^2_{H^{-1}} \\ &\quad + \lambda_{\max}\left(H^{\frac{1}{2}}E'(t)H^{-\frac{1}{2}} + H^{-\frac{1}{2}}E(t)H^{\frac{1}{2}}\right)\|x\|^2_{H^{-1}} + \epsilon^\star e^{-\beta t}\end{aligned} \tag{17}$$

Since the matrix $\left(H^{\frac{1}{2}}E'(t)H^{-\frac{1}{2}} + H^{-\frac{1}{2}}E(t)H^{\frac{1}{2}}\right)$ is a symmetric matrix, then all its eigenvalues are reals. Using result of theorem 1, then we can replace $\lambda_{\max}\left(H^{\frac{1}{2}}E'(t)H^{-\frac{1}{2}} + H^{-\frac{1}{2}}E(t)H^{\frac{1}{2}}\right)$ by $\mu\left(H^{\frac{1}{2}}E'(t)H^{-\frac{1}{2}} + H^{-\frac{1}{2}}E(t)H^{\frac{1}{2}}\right)$. Using result of theorem 2, we obtain

$$\begin{aligned}&\mu\left(H^{\frac{1}{2}}E'(t)H^{-\frac{1}{2}} + H^{-\frac{1}{2}}E(t)H^{\frac{1}{2}}\right) \\ &\leq \mu\left(\left|H^{\frac{1}{2}}E'(t)H^{-\frac{1}{2}} + H^{-\frac{1}{2}}E(t)H^{\frac{1}{2}}\right|\right) \\ &\leq \mu\left(\left|H^{\frac{1}{2}}\right|\left|E'(t)\right|\left|H^{-\frac{1}{2}}\right| + \left|H^{-\frac{1}{2}}\right|\left|E(t)\right|\left|H^{\frac{1}{2}}\right|\right) \\ &\leq \epsilon\mu\left(\left|H^{\frac{1}{2}}\right|W'\left|H^{-\frac{1}{2}}\right| + \left|H^{-\frac{1}{2}}\right|W\left|H^{\frac{1}{2}}\right|\right)\end{aligned} \tag{18}$$

Finally, we conclude that if the parameter γ is chosen to satisfy

$$\begin{aligned}&\gamma + \lambda_{\min}\left(H^{-\frac{1}{2}}BB'H^{\frac{1}{2}}\right) > \\ &\epsilon\mu\left(\left|H^{\frac{1}{2}}\right|W'\left|H^{-\frac{1}{2}}\right| + \left|H^{-\frac{1}{2}}\right|W\left|H^{\frac{1}{2}}\right|\right),\end{aligned} \tag{19}$$

then $\|x\|$ converges asymptotically to zero.

Remark 1 *The fact that W (resp. W') is left multiplied by $\left|H^{-\frac{1}{2}}\right|$(resp. $\left|H^{\frac{1}{2}}\right|$) and right multiplied by $\left|H^{\frac{1}{2}}\right|$(resp. $\left|H^{-\frac{1}{2}}\right|$), increases the chance to obtain*

$$\mu\left(\left|H^{\frac{1}{2}}\right| W' \left|H^{-\frac{1}{2}}\right| + \left|H^{-\frac{1}{2}}\right| W \left|H^{\frac{1}{2}}\right|\right) < \gamma.$$

Application to missile autopilot model

Here, we consider the state space model of a missile autopilot described as

$$\begin{bmatrix} \dot{q} \\ \dot{\alpha} \\ \dot{\eta} \end{bmatrix} = \underbrace{\begin{bmatrix} a_{1,1}(Mc,\, h) & a_{1,2}(Mc,\, h) & a_{1,3}(Mc,\, h) \\ 1 & a_{2,2}(Mc,\, h) & a_{2,3}(Mc,\, h) \\ 0 & 0 & -30 \end{bmatrix}}_{A+E(t)} \begin{bmatrix} q \\ \alpha \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 30 \end{bmatrix} (u + \xi(t)),$$

where $a_{1,1}(Mc,\, h)$, $a_{1,2}(Mc,\, h)$, $a_{1,3}(Mc,\, h)$, $a_{2,2}(Mc,\, h)$, $a_{2,3}(Mc,\, h)$ are the uncertain elements that depend upon the altitude h and the Mach number Mc. The elements of the state vector are: the pitch rate q, the angle of attack α, and the elevator deflection angle η. We suppose that $\xi(t)$ is a bounded disturbance that affects the controller u. Within an altitude of 10000 m and $Mc \geq 2$, the nominal matrices of the missile dynamics are[3]

$$A = \begin{bmatrix} -1.364 & -92.82 & -128 \\ 1 & -4.68 & -0.087 \\ 0 & 0 & -30 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 30 \end{bmatrix}, \quad (20)$$

$$E(t) = \begin{bmatrix} 1.0310\sin(t) & 0.42\cos(t) & 5.32\sin(t) \\ 0 & 0.4\sin(2\,t) & 0.370\cos(t) \\ 0 & 0 & 0 \end{bmatrix}. \quad (21)$$

Note that we could bound the uncertain matrix $|E(t)|$ by $1.12\,I$. i.e., $\epsilon = 1.12$ and $W = I$. For $\gamma = 100$, we obtain

$$H = \begin{bmatrix} 104.7558841 & -1.854941671 & 41.14908488 \\ -1.854941671 & 0.03977807201 & -0.5999936448 \\ 41.14908488 & -0.5999936448 & 22.50000000 \end{bmatrix} > 0. \quad (22)$$

The set of the eigenvalues of H is

$$\left\{.4315607889\,10^{-2}, 5.453752292, 121.8375943\right\}.$$

We have

$$H^{-1} = \begin{bmatrix} 0.1865720776 & 5.944690050 & -0.1826886227 \\ 5.944690050 & 231.4688281 & -4.699499094 \\ -0.1826886227 & -4.699499094 & 0.2532355577 \end{bmatrix}, \quad (23)$$

and

$$H^{\frac{1}{2}} = \begin{bmatrix} 9.7595 & -0.1865 & 3.0779 \\ -0.1865 & 0.0703 & -0.0071 \\ 3.0779 & -0.0071 & 3.6092 \end{bmatrix}. \quad (24)$$

Since

$$\mu_2\left(\left|H^{\frac{1}{2}}\right| W' \left|H^{-\frac{1}{2}}\right| + \left|H^{\frac{1}{2}}\right| W \left|H^{-\frac{1}{2}}\right|\right) = 51.8313 < \gamma = 100.$$

then the condition of stability is verified and the asymptotic convergence of the states is guaranteed.

Conclusion

A simple controller with a new non restrictive condition of quadratic stability has been developed. The controller is designed to overcome both uncertain dynamics, due to model imprecision, and external perturbation coming resisting to the system actuators. The controller is basically designed to be independent from the types or the forms of uncertainties. This property enlarges the field of application of the developed controller and makes the user free from the usual matching conditions, generally encountered in such situations. Computation of the controller gain is fulfilled by the resolution of a parameter-dependent Lyapunov-like equation and the controller gain is based upon the knowledge of the upper bounds of uncertainties. The controller design algorithm is quite simple and the computational requirements is drastically reduced to a simple matrix computation.

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