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Brief paper

Linear time-derivative trackers[☆]

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Abstract

The design of an ideal differentiator is a difficult and a challenging task. In this paper we discuss the properties and the limitations of two different structures of linear differentiation systems. The first time-derivative observer is formulated as a high-gain observer where the observer gain is calculated through a Lyapunov-like dynamical equation. The second one is given in Brunovski form in which the output to be differentiated appears as a control input and the differentiation gain is calculated from the dual Lyapunov equation of the first differentiation observer. A discrete-time version of the second form is given. Finally, illustrative examples are presented to show their strengths and weaknesses.

Keywords: Differentiation; Lyapunov theory; Time-varying systems; Discrete-time systems; Observers

1. Introduction

The necessity to evaluate the time-derivative of signals arises frequently in many areas of research (Lanshammar, 1982; Mahapatra, 2000; Cullum, 1971; Craven & Wahba, 1979; Ibrir, 2000). In human motion analysis the determination of internal forces and moments requires estimation of body segment acceleration, which is obtained by double differentiation of displacement data. Thermal industrial applications includes estimation of heating rates from temperature measurements. In target tracking and radar applications the need of velocity estimation from measured position data still a difficult task and a challenging problem (Wong, 2000). In the last years, high-gain observers have served as a robust and efficient tool for semi-global stabilization and observation of nonlinear systems (Teel & Praly, 1995; Atassi & Khalil, 1999; Ljung & Gald, 1994; Ibrir & Diop, 1999). Even though the development of various techniques, the time-derivative estimation still have real difficulties and necessitates more attention in order to be reliably used in practice (Lanshammar, 1982; Usui & Amidror, 1982; Allum, 1975; Tornambè, 1992; Dabroom & Khalil, 1999; Ibrir, 1999, 2000, 2001; Ibrir & Diop, 1999).

In Ibrir (2000, 2001), the author proposes a time-varying linear system to estimate the first $(n-1)$th derivative of any bounded signal $y(t)$. The state-space representation of this system is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + H^{-1}(t)C'(y(t) - Cx(t)), \\ \dot{H}(t) &= -\gamma H(t) - A'H(t) - H(t)A + C'C, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $y(t)$ is a continuously differentiable bounded function and $A \in \mathbb{R}^{n\times n} : A_{i,j} = \delta_{i,j-1}, 1 \leqslant i, j \leqslant n$, $C \in \mathbb{R}^{1\times n} : C_i = \delta_{i,1}, 1 \leqslant i \leqslant n$ are given in observable canonical form. Here, $\delta_{i,j}$ stands for the Kronecker symbol. Time-varying system (1) converges, in finite time, to a linear system of the following form:

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t) + C_n^i \gamma^i (y(t) - x_1(t)), \quad 1 \leqslant i \leqslant n-1 \\ \dot{x}_n(t) &= C_n^n \gamma^n (y(t) - x_1(t)), \end{aligned} \tag{2}$$

where C_n^i is the binomial coefficient. In this paper, we shall call the differentiators having the form of system (1) or (2), *the observable canonical form differentiation observers*. This nomenclature comes from the fact that (A, C) is an observable pair. Differentiator (2) is exactly the high-gain differentiation observer studied by Esfandiari and Khalil

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(1989), Dabroom and Khalil (1999) and Tornambè (1992), where γ is replaced by $1/\varepsilon$ and C_n^k by some constant α_k.

The main purpose of designing such differentiation systems is the ability to estimate the higher derivatives of $y(t)$ without any knowledge of its dynamics. However, this ill-posed problem renders the trade-off between the desired performances an extremely difficult task. The main focus of this paper is to investigate how to append the performances of linear differentiation systems in terms of peaking, differentiation error and noise filtering. First, we show how the peaking phenomenon can be removed in high-gain differentiation systems, written in observable canonical form. Subsequently, we propose another form of high-gain differentiators, given in Brunovski form, in which the output to be differentiated appears as a control input and the differentiation gain is calculated from the dual Lyapunov matrix equation of the first differentiation observer (1). For this class of controllable differentiation systems, we show that noise filtering can be improved by increasing the differentiation order n. Practical issue to reduce the peaking phenomenon is also discussed. The strength, the weakness and the common performances of the two new kinds of differentiation observers will be highlighted on the scope of similar existing differentiators. Finally, a discrete version of the differentiation observer is included. Throughout this paper, we note $\mathbb{R}$ and $\mathbb{R}_{>0}$: the set of real numbers and the set of real positive numbers, respectively. $\mathbb{Z}$ is the set of integer numbers. We note

$$\|f(t)\|_\infty = \sup_{t\geqslant 0}|f(t)|,\ \|f(t)\|_1 = \int_0^\infty |f(t)|\,\mathrm{d}t.$$

$\|A\|_2 = \max\{\sqrt{\lambda}:\lambda$ is the eigenvalue of $A^TA\}$, $\mathrm{eig}(A)$ is the set of the eigenvalues of the matrix A. $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{i,j}|\ \lambda_{\min}(A)$: is the smallest eigenvalue of A. $\lambda_{\max}(A)$: is the largest eigenvalue of A. $\mathscr{S}^+(n,\mathbb{R})$ denotes the set of positive-definite matrices of order n. $\mathscr{L}f(t)$ stands for the Laplace transform of the function $f(t)$. $y^{(i)}(t)$ represents the ith derivative of $y(t)$. $y_k^{(i)}$ denotes the ith derivative of $y(t)$ at time $t=t_k$ $\mu(A)=\lambda_{\max}((A+A')/2)$ is the measure of the matrix A, and I_d is the identity matrix of appropriate dimension. $(\cdot * \cdot)$ is the usual convolution operator.

2. The peaking phenomenon

Differentiation systems proposed in references Tornambè (1992), Dabroom and Khalil (1999) suffer from a serious drawback, the peaking phenomenon. Detailed discussions of this phenomenon are given in [Sussmann and Kokotovic (1991) and Dabroom and Khalil (1999)]. In this section, we show how the user can make system (1) a nonpeaking differentiator with a suitable choice of the initial matrix $H(0)$. System (1) behaves as a stable time-varying linear system controlled by the input $H^{-1}(t)C'y(t)$. Putting $y(t)=0$ into system (1) gives

$$\begin{aligned}\dot{x}(t) &= (A - H^{-1}(t)C'C)x(t),\\ \dot{H}(t) &= -\gamma H(t) - A'H(t) - H(t)A + C'C.\end{aligned}\tag{3}$$

The explicit solution of $H(t)$ is given by

$$H(t) = \mathrm{e}^{-\gamma t}\mathrm{e}^{-A't}H(0)\mathrm{e}^{-At} + \int_0^t \mathrm{e}^{-\gamma(t-\tau)}\mathrm{e}^{-A'(t-\tau)}C'C\mathrm{e}^{-A(t-\tau)}\,\mathrm{d}\tau.\tag{4}$$

If we take $V(x(t)) = x'(t)H(t)x(t)$ as a Lyapunov function candidate to (3), then one can easily show that $\dot{V}(x(t)) \leqslant -\gamma V(x(t))$ which implies that $V(x(t)) \leqslant \mathrm{e}^{-\gamma t}V(0)$, or

$$\|x(t)\|^2 \leqslant (\mathrm{e}^{-\gamma t}\|H(0)\|\|x(0)\|^2)/\lambda_{\min}(H(t)).\tag{5}$$

Using the properties of symmetric and positive-definite matrices, we have

$$\lambda_{\min}(H(t)) \geqslant \lambda_{\min}(\mathrm{e}^{-\gamma t}\mathrm{e}^{-A't}H(0)\mathrm{e}^{-At}) + \lambda_{\min}\left(\int_0^t \mathrm{e}^{-\gamma(t-\tau)}\mathrm{e}^{-A'(t-\tau)}C'C\mathrm{e}^{-A(t-\tau)}\,\mathrm{d}\tau\right).\tag{6}$$

Since (A,C) is observable, then there exists an $\varepsilon > 0$ such that

$$\lambda_{\min}\left(\int_0^t \mathrm{e}^{-\gamma(t-\tau)}\mathrm{e}^{-A'(t-\tau)}C'C\mathrm{e}^{-A(t-\tau)}\,\mathrm{d}\tau\right) \geqslant \varepsilon \quad \forall t > 0.$$

Moreover,

$$\lambda_{\min}(\mathrm{e}^{-\gamma t}\mathrm{e}^{-A't}H(0)\mathrm{e}^{-At}) \geqslant \lambda_{\min}(\mathrm{e}^{-\gamma t}I_d)\lambda_{\min}(\mathrm{e}^{-A't}H(0)\mathrm{e}^{-At}).$$

Then

$$\lambda_{\min}(H(t)) \geqslant \mathrm{e}^{-\gamma t}\lambda_{\min}(\mathrm{e}^{-A't}H(0)\mathrm{e}^{-At}) + \varepsilon.\tag{7}$$

If we choose $H(0) = 1/\varepsilon_0\, I_d$, then

$$\lambda_{\min}(\mathrm{e}^{-A't}H(0)\mathrm{e}^{-At}) \geqslant \frac{1}{\varepsilon_0}\lambda_{\min}(\mathrm{e}^{-A't}\mathrm{e}^{-At}) \geqslant \frac{1}{\varepsilon_0}\mathrm{e}^{-2\mu(-A)t} = \frac{1}{\varepsilon_0}\mathrm{e}^{-t}.$$

Finally,

$$\|x(t)\|^2 \leqslant \frac{\mathrm{e}^{-\gamma t}}{\mathrm{e}^{-(\gamma+1)t} + \varepsilon\varepsilon_0}\|x(0)\|^2.\tag{8}$$

When the value of γ is high, the function $\mathrm{e}^{-\gamma t}/(\mathrm{e}^{-(\gamma+1)t} + \varepsilon\varepsilon_0)$ is close to 1. Consequently, the peaking phenomenon does not appear with increasing values of γ. Taking the Laplace transform of (2), we have

$$\left[\frac{X_\infty(s)}{Y(s)}\right]_i = s^{i-1}\left(1 - \frac{\sum_{k=0}^{i-1} C_n^k\gamma^k s^{n-k}}{(s+\gamma)^n}\right),\quad 1 \leqslant i \leqslant n\tag{9}$$

such that $X(s)$ and $Y(s)$ are the Laplace transforms of the vector $x(t)$ and the signal $y(t)$, respectively. This gives

$$y^{(i-1)}(t) - x_i(t) = \rho_i(t) * y^{(i)}(t),\quad 1 \leqslant i \leqslant n,\tag{10}$$

where

$$\rho_i(t) = \mathscr{L}^{-1}\left(\sum_{k=0}^{i-1} C_n^k \gamma^k s^{n-k}/(s+\gamma)^n\right).$$

Using Young's inequality, we obtain

$$\begin{aligned} \left|\left|x_i(t) - y^{(i-1)}(t)\right|\right|_\infty &\leqslant ||\rho_i(t)||_1 \left|\left|y^{(i)}(t)\right|\right|_\infty \\ &= \frac{\kappa_i \|y^{(i)}(t)\|_\infty}{\gamma}, \\ & 1 \leqslant i \leqslant n,\ \kappa_i \in \mathbb{R}_{>0}. \end{aligned} \tag{11}$$

As we have mentioned before, observer (1) is able to estimate the higher-derivatives of the output $y(t)$ without any knowledge of the dynamics of $y(t)$. Moreover, the differentiation error can be handled by varying the parameter γ. To show the usefulness of differentiator (1), let us consider the nonlinear system

$$\begin{aligned} \dot{\xi}_1(t) &= \xi_2(t) - \xi_1^3(t), \\ \dot{\xi}_2(t) &= -\xi_1(t) - \xi_2^3(t), \\ y(t) &= \xi_1(t). \end{aligned} \tag{12}$$

From the first equation of (12), we have $\xi_2(t) = \dot{y}(t) + y^3(t)$. To observe the state $\xi_2(t)$, it is sufficient to construct an observer to estimate $\dot{y}(t)$. Since system (12) is asymptotically stable, then $\xi_1(t)$ and $\xi_2(t)$ are uniformly bounded. Then for γ sufficiently large, a *nonpeaking observer* is readily constructed as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + H^{-1}(t)C'(y(t) - Cx(t)), \\ \dot{H}(t) &= -\gamma H(t) - A'H(t) - H(t)A + C'C, \\ \hat{\xi}_2(t) &= x_2(t) + y^3(t), \end{aligned} \tag{13}$$

where $A \in \mathbb{R}^{2\times 2}$, $C \in \mathbb{R}^{1\times 2}$ are defined as in (1) and $H(0)^{-1} = \varepsilon I_d$ $(0 < \varepsilon \ll 1)$. Remark that the high-gain term $H^{-1}(t)C'(y(t) - Cx(t))$ in (13) is not used to oppose to the adverse nonlinearities of system (12). Hence, the absent information $\dot{y}(t)$ is totally constructed without any information of the dynamics (12). Therefore, the observation strategy (1) can be considered as an alternative to Luenberger high-gain observers, see for example Tornambè (1992) and Gauthier, Hammouri, and Othman (1992). The main disadvantage of this technique is the loss of the asymptotic convergence of the observer, but in practice it is sufficient to maintain a small estimation error by increasing the value of γ, see Eq. (11).

Even though differentiator (1) offers a nice transient behavior and a free adjustable differentiation error, see (11), its sensitivity to noise is important because the high-gain parameters appear in the whole differential equations of the differentiator. The aim of the next section is to develop a new differentiator that preserves the properties of (1) and behaves more resistant to noise. The new differentiator will be in the dual form of (1), where the signal $y(t)$, to be differentiated, does not appear in the first equation of the state space representation except in the last equation, where it appears as a control input.

3. Differentiators in controllable canonical form

3.1. The continuous-time derivative tracker

The design strategy is given in the next theorem.

Theorem 1. *Consider the time-varying linear system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BB'P^{-1}(t)(x(t) - C'y(t)), \\ \dot{P}(t) &= -\gamma P(t) - P(t)A' - AP(t) + BB', \end{aligned} \tag{14}$$

where $x(t)\colon \mathbb{R}_{\geqslant 0} \mapsto \mathbb{R}^n$ is the state vector, $y(t)\colon \mathbb{R}_{\geqslant 0} \mapsto \mathbb{R}$ is a smooth bounded signal along with its higher derivatives. Then for large values of γ, each state $x_i(t)$ approximates the $(i-1)$th derivative of the input signal $y(t)$ when $t \to \infty$. The nominal matrices of system (14) are

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{n\times n}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1},$$

$$C = [1\ 0\ \dots\ 0]_{1\times n}. \tag{15}$$

The proof of Theorem 1 necessitates the result of the following lemma.

Lemma 1. *For any $P(0) \in \mathscr{S}^+(n, \mathbb{R})$, the matrix $P(t)$ converges to the unique positive-definite P_∞ defined as $[P_\infty]_{i,j} = p_{i,j}/\gamma^{2n-i-j+1}, 1 \leqslant i, j \leqslant n$, and $p_{i,j}$ is a real constant which do not depend on γ.*

Proof. Let $A_\gamma = -(\gamma/2)I_{\mathrm{d}} - A'$. Then $P(t)$ verifies the Lyapunov matrix equation

$$\dot{P}(t) = A'_\gamma P(t) + P(t)A_\gamma + BB'. \tag{16}$$

The solution of the matrix differential (16) is

$$\begin{aligned} P(t) &= \mathrm{e}^{A'_\gamma t}P(0)\mathrm{e}^{A_\gamma t} + \int_0^t \mathrm{e}^{A'_\gamma(t-\tau)}BB'\mathrm{e}^{A_\gamma(t-\tau)}\,\mathrm{d}\tau \\ &= \mathrm{e}^{-\gamma t}\mathrm{e}^{-At}P(0)\mathrm{e}^{-A't} \\ &\quad + \int_0^t \mathrm{e}^{-\gamma(t-\tau)}\mathrm{e}^{-A(t-\tau)}BB'\mathrm{e}^{-A'(t-\tau)}\,\mathrm{d}\tau. \end{aligned} \tag{17}$$

Since the first term of (17) vanishes to zero when times elapses, then

$$P_\infty = \int_0^\infty \mathrm{e}^{-\gamma(t-\tau)}\mathrm{e}^{-A(t-\tau)}BB'\mathrm{e}^{-A'(t-\tau)}\,\mathrm{d}\tau. \tag{18}$$

Since $A^k = 0$ for $k \geqslant n$ then

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k = \sum_{k=0}^{n-1} \frac{A^k}{k!} t^k.$$

Whatever the dimension of A, we have $[A]_{i,j} = \delta_{i,j-1}$. This yields

$$[e^{-A(t-\tau)}]_{i,j} = \delta_{i,j} + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \delta_{i,j-k}(t-\tau)^k,$$

and

$$[e^{-A'(t-\tau)}]_{i,j} = \delta_{i,j} + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \delta_{i-k,j}(t-\tau)^k.$$

Consequently;

$$[e^{-A(t-\tau)}BB']_{i,j} = \delta_{n,j}\delta_{i,j} + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \delta_{i,j-k}\delta_{n,j}(t-\tau)^k.$$

We get

$$[e^{-A(t-\tau)}BB'e^{-A'(t-\tau)}]_{i,j} = \frac{(-1)^{2n-i-j}}{(n-i)!(n-j)!}(t-\tau)^{2n-i-j}. \tag{19}$$

Finally, for $1 \leqslant i,j \leqslant n$

$$\lim_{t\to\infty}[P(t)]_{i,j} = \lim_{t\to\infty} \frac{(-1)^{2n-i-j}}{(n-i)!(n-j)!} \times \int_0^{\infty} e^{-\gamma(t-\tau)}(t-\tau)^{2n-i-j}\,d\tau. \tag{20}$$

Using

$$\lim_{t\to\infty}\int_0^{\infty} e^{-\gamma(t-\tau)}(t-\tau)^{2n-i-j}\,d\tau = \frac{(2n-i-j)!}{\gamma^{2n-i-j+1}},$$

then

$$[P_\infty]_{i,j} = (-1)^{2n-i-j}\frac{(2n-i-j-2)!}{(n-i)!(n-j)!}\frac{1}{\gamma^{2n-i-j+1}} = (-1)^{2n-i-j}\frac{C_{2n-i-j}^{n-i}}{\gamma^{2n-i-j+1}}. \tag{21}$$

This yields

$$[\tilde{P}_\infty]_{i,j} = (-1)^{2n-i-j}C_{2n-i-j}^{n-i}, \quad 1 \leqslant i,j \leqslant n, \tag{22}$$

where $\tilde{P}_\infty$ is the solution of the matrix algebraic equation $-\tilde{P}_\infty - \tilde{P}_\infty A' - A\tilde{P}_\infty + BB' = 0$. Consequently,

$$[P_\infty^{-1}]_{i,j} = [\tilde{P}_\infty^{-1}]_{i,j}\gamma^{2n-i-j+1}, \quad 1 \leqslant i,j \leqslant n. \qquad \square \tag{23}$$

Proof of theorem 1. Consider the linear system

$$\dot{\hat{x}}(t) = A\hat{x}(t) - BB'P_\infty^{-1}(\hat{x}(t) - C'y(t)),$$

$$\gamma P_\infty + P_\infty A' + AP_\infty - BB' = 0. \tag{24}$$

Then for any bounded signal $y(t) \in \mathscr{C}^\infty$ we shall prove that $\lim_{t\to\infty}(x_i(t) - \hat{x}_i(t)) = 0$, $1 \leqslant i \leqslant n$, where $x(t)$ is the state vector of system (14). Note $e(t) = x(t) - \hat{x}(t)$, then

$$\dot{e}(t) = Ae(t) - BB'P^{-1}(t)x(t) + BB'P^{-1}(t)C'y(t) + BB'P_\infty^{-1}\hat{x}(t) - BB'P_\infty^{-1}C'y(t). \tag{25}$$

By adding and subtracting the term $BB'P_\infty^{-1}(x(t) - \hat{x}(t))$, the last equation becomes

$$\dot{e}(t) = (A - BB'P_\infty^{-1})e(t) + BB'(P_\infty^{-1} - P^{-1}(t)) \times (x(t) - C'y(t)). \tag{26}$$

Now, let us add and subtract the term $BB'(P_\infty^{-1} - P^{-1}(t))\hat{x}(t)$ from the right-hand side of the last equation, we obtain

$$\dot{e}(t) = (A - BB'P_\infty^{-1})e(t) + BB'(P_\infty^{-1} - P^{-1}(t))e(t) + BB'(P_\infty^{-1} - P^{-1}(t))(\hat{x}(t) - C'y(t)). \tag{27}$$

Let us assign $V(e(t)) = e'(t)P_\infty^{-1}e(t)$ as a Lyapunov function to system (27). Then

$$\begin{aligned}\dot{V} &= \dot{e}'(t)P_\infty^{-1}e(t) + e'(t)P_\infty^{-1}\dot{e}(t)\\ &= e'(t)(-\gamma P_\infty^{-1} - P_\infty^{-1}BB'P_\infty^{-1})e(t) + 2e'(t)\\ &\quad (P_\infty^{-1} - P^{-1}(t))BB'P_\infty^{-1}e(t)\\ &\quad +2(\hat{x}(t) - C'y(t))'(P_\infty^{-1} - P^{-1}(t))BB'P_\infty^{-1}e(t).\end{aligned} \tag{28}$$

This gives

$$\dot{V} \leqslant -\gamma V + 2\|P_\infty\|\|P_\infty^{-1} - P^{-1}(t)\|V + 2\|\hat{x}(t) - C'y(t)\|\|P_\infty^{-1} - P^{-1}(t)\|\sqrt{V}. \tag{29}$$

We have

$$\|P_\infty^{-1}\| \leqslant \sqrt{n}\|P_\infty^{-1}\|_\infty = c_1\gamma^{2n-1}, \|P_\infty\| \leqslant \sqrt{n}\|P_\infty\|_\infty = \frac{c_2}{\gamma}, \lambda_{\min}(P_\infty) = \frac{1}{\|P_\infty^{-1}\|} \leqslant \frac{c_{\min}}{\gamma^{2n-1}}, \lambda_{\max}(P_\infty) \leqslant \frac{c_{\max}}{\gamma},$$

where c_1, c_2, $c_{\min}$, $c_{\max}$ are real constants which depend on n. Since (A,B) is a controllable pair, then for $t \geqslant 0$, the controllability Gramian is always positive, and hence, $P(t)$ is always positive definite, see Eq. (17). From Eq. (17), we deduce the explicit solution of the difference

$$P(t) - P_\infty = e^{-\gamma t}e^{-At}(P(0) - P_\infty)e^{-A't}. \tag{30}$$

Then

$$\|P(t) - P_\infty\| \leqslant \|P(0) - P_\infty\|\|e^{-At}e^{-A't}\|e^{-\gamma t} \leqslant \|P(0) - P_\infty\|\lambda_{\max}(e^{-At}e^{-A't})e^{-\gamma t}. \tag{31}$$

Using inequality

$$\lambda_{\max}(e^{-At}e^{-A't}) \leqslant e^{2\mu(A')t} = e^{\lambda_{\max}\{(A+A')/2\}t} = e^t,$$

then

$$\|P(t) - P_\infty\| \leqslant C_0 e^{-(\gamma-1)t}, \tag{32}$$

where $C_0 = \|P(0) - P_\infty\|$. Consequently, the difference $P_\infty^{-1} - P^{-1}(t)$ can be bounded as follows:

$$\begin{aligned}\|P_\infty^{-1} - P^{-1}(t)\| &= \|P^{-1}(t)\|\|(P(t) - P_\infty)\|\|P_\infty^{-1}\| \\ &\leqslant \|(P(t) - P_\infty)\|\|P_\infty^{-1}\|^2 \\ &\leqslant C_0 c_1 \gamma^{2(2n-1)} e^{-(\gamma-1)t}.\end{aligned} \tag{33}$$

Furthermore, for $t \geqslant 0$, we have

$$\hat{x}(t) = \mathrm{e}^{(A-BB'P_\infty^{-1})t}\|\hat{x}(0)\| + \int_0^t \mathrm{e}^{(A-BB'P_\infty^{-1})(t-\tau)} BB'P_\infty^{-1}C'y(\tau)\,\mathrm{d}\tau. \tag{34}$$

The matrix $A - BB'P_\infty^{-1}$ is Hurwitz, one could easily show that

$$\begin{aligned}&(A - BB'P_\infty^{-1})'P_\infty^{-1} + P_\infty^{-1}(A - BB'P_\infty^{-1}) \\ &\quad = -\gamma P_\infty^{-1} - P_\infty^{-1}BB'P_\infty^{-1} < 0.\end{aligned} \tag{35}$$

We conclude that $\int_0^\infty \mathrm{e}^{(A-BB'P_\infty^{-1})\tau}\,\mathrm{d}\tau = -(A - BB'P_\infty^{-1})^{-1}$. For $1 \leqslant i, j \leqslant n$, we have

$$(A - BB'P_\infty^{-1})_{i,j}^{-1} = \begin{cases} 1 & \textit{if } j = i-1, \\ -C_n^j/\gamma^j & \textit{if } i = 1, \\ 0 & \text{else.}\end{cases} \tag{36}$$

This gives $\|(A - BB'P_\infty^{-1})^{-1}\| \leqslant C_1(\gamma)$, where $C_1(\gamma) = \sqrt{n}\max\{1, \sum_{i=1}^n C_n^i/\gamma^i\}$. The solution $\hat{x}(t)$ can be bounded as follows:

$$\begin{aligned}\|\hat{x}(t)\| &\leqslant K(\gamma)\|\hat{x}_0\|\mathrm{e}^{-\gamma t} + \|(A - BB'P_\infty^{-1})^{-1}\|\|P_\infty^{-1}\|\|y(t)\|_\infty \\ &\leqslant K(\gamma)\|\hat{x}_0\|\mathrm{e}^{-\gamma t} + C_2(\gamma)\gamma^{2n-1},\end{aligned} \tag{37}$$

where $C_2(\gamma) = c_1 C_1(\gamma)\|y(t)\|_\infty$ and $K(\gamma)$ is some constant which depends on γ. Let $W = \sqrt{V}$. Using (37), (32), (33), then inequality (29) becomes $\dot{W} \leqslant -((\gamma/2) - C_3(\gamma)\mathrm{e}^{-(\gamma-1)t})W + C_4(\gamma)\mathrm{e}^{-(\gamma-1)t}$, where $C_3(\gamma) = C_0 c_2^2 \gamma^{4n-3}$ and $C_4(\gamma) = C_0 c_1 (K(\gamma)\|\hat{x}_0\| + C_2(\gamma)\gamma^{2n-1})\gamma^{2(2n-1)}$. Finally,

$$\begin{aligned}W(t) \leqslant &\left[W(0) + \int_0^t C_4(\gamma)\exp\frac{-(1/2)\gamma^2\tau + (3/2)\gamma\tau - \tau + C_3(\gamma)\mathrm{e}^{-(\gamma-1)\tau}}{\gamma-1}\right. \\ &\left.\times\,\mathrm{d}\tau\right]\exp\frac{-(1/2)\gamma^2 t + (1/2)\gamma t - C_3(\gamma)\mathrm{e}^{-(\gamma-1)t}}{\gamma-1}.\end{aligned} \tag{38}$$

We deduce that $\lim_{t\to\infty}\hat{x}(t) - x(t) = 0$. Taking Laplace transform of (24), we obtain for $(1 \leqslant i \leqslant n)$

$$\frac{\hat{X}_i(s)}{Y(s)} = \frac{\gamma^n s^{i-1}}{(s+\gamma)^n}. \tag{39}$$

It is clear that $\lim_{\gamma\to\infty}\hat{X}_i(s)/Y(s) = \lim_{\gamma\to\infty}\gamma^n s^{i-1}/(s+\gamma)^n = s^{i-1}$, it means that $\hat{x}_i(t)$ approximates the derivative $y^{(i-1)}(t)$ for $2 \leqslant i \leqslant n$. This ends the proof of Theorem 1. □

Finally, the state-space representation of the differentiator is

$$\begin{aligned}&\dot{x}_i(t) = x_{i+1}(t), \quad 1 \leqslant i \leqslant n-1, \\ &\dot{x}_n(t) = -\gamma^n(x_1(t) - y(t)) - \sum_{i=1}^{n-1} C_n^i \gamma^i x_{n-i+1}(t).\end{aligned} \tag{40}$$

3.2. Discussion

3.2.1. Comparative study and basic properties

One of the elegant properties of differentiator system (14) is the dependency of differentiation error to the only one tuning parameter γ. From Eq. (39), we have

$$\hat{X}_i(s) - s^{i-1}Y(s) = \frac{1}{s}\left(\frac{\gamma^n}{(s+\gamma)^n} - 1\right)s^i Y(s), \quad 1 \leqslant i \leqslant n. \tag{41}$$

which gives the error bound

$$\left\|\hat{x}_i(t) - y^{(i-1)}(t)\right\|_\infty \leqslant \frac{n}{\gamma}\left\|y^{(i)}(t)\right\|_\infty, \quad 1 \leqslant i \leqslant n. \tag{42}$$

By comparison of (42) and (11), we conclude that differentiators (14) and (1) offer a similar differentiation error when time elapses. From Eq. (39), we see that the analog differentiation is achieved over a limited frequency range γ. Each state $x_i(t)$ of differentiator (39) is the output of a concatenated ideal $(i-1)$th-order differentiator and a low-pass filter of order n. By increasing the differentiation order n, noise is more attenuated, but bandwidth becomes smaller than the usual one. The controllable canonical form (40) seems to be so interesting when the differentiator is used in closed-loop configurations. In addition, we see that $y(t)$ just appears in the last equation, so a great amount of eventual additive noise shall be eliminated because of the presence of the successive n integrators.

3.2.2. The peaking phenomenon

Unfortunately, the smooth variation of the differentiation gain in (14) does not remove the peaking phenomenon as in (1) but it can be reduced by choosing $P^{-1}(0) = \varepsilon I$, where ε is a small positive parameter. Another way to reduce the peaking phenomenon is to consider system (40) with

$$\gamma = \gamma(t) = \begin{cases}\mu t & \text{if } 0 \leqslant t \leqslant t_{\max}, \\ \mu t_{\max} & \text{otherwise,}\end{cases} \tag{43}$$

where μ and $t_{\max}$ are chosen according to the desired maximum error that depends on the value of $\gamma_{\max} = \mu t_{\max}$. For $0 \leqslant t \leqslant t_{\max}$, the dynamic equations of the differentiator are

$$\begin{aligned}&\dot{x}_i(t) = x_{i+1}(t), \quad 1 \leqslant i \leqslant n-1 \\ &\dot{x}_n(t) = -(\mu t)^n(x_1(t) - y(t)) \\ &\qquad - \sum_{i=1}^{n-1} C_n^i(\mu t)^i x_{n-i+1}(t).\end{aligned} \tag{44}$$

For $y(t)=0$, the last system is asymptotically stable since

$$x_i(t)=\frac{d^{i-1}}{dt^{i-1}}\left(e^{-(\mu/2)t^2}\left(\sum_{i=1}^{n}C_i e^{\lambda_i t}\right)\right),\ \lambda_i, C_i\in\mathbb{R},$$

$$1\leqslant i\leqslant n,$$

is the unique solution of (44). Since $y(t)$ is uniformly bounded, then for $0\leqslant t\leqslant t_{\max}$, the states of the differentiator cannot escape to infinity. By increasing $t_{\max}$ the precision of the derivative estimates can be considerably improved without changing the transient behavior of the differentiator. This powerful technique allow us to handle the trade-off between the differentiation error, the peaking rate, and the filtering of the derivative estimates.

4. The discrete-time derivative tracker

In practice derivative estimation is the process of inferring values of higher derivative at a specific instant of time from indirect, inaccurate and uncertain observations. The objective of this section is to present the discrete-time version of observer (14). The whole design of discrete-time differentiator is given in the next theorem.

Theorem 2. *Consider the discrete-time system*

$$\begin{aligned}x_{k+1}&=e^{A\delta}x_k-\delta BB'P_k^{-1}(e^{A\delta}x_k-C'y_k),\\ P_{k+1}&=\lambda e^{-A\delta}P_k e^{-A'\delta}+\delta BB',\end{aligned}\tag{45}$$

where $x_k\in\mathbb{R}^n$ is the state vector and $(y_k)_{k\in\mathbb{Z}_{\geqslant 0}}$ is a uniformly bounded signal such that $\sup_{k\geqslant 0}|y_k^{(i)}|<\infty$ for all i. Then for all $\lambda,(0<\lambda<1)$ such that

$$\mathrm{eig}(\sqrt{\lambda}e^{-A\delta})<1,\tag{46}$$

the state vector x_k estimates the derivative vector $[y_k\ \dot{y}_k\ \ddot{y}_k\ \cdots\ y_k^{(n-1)}]'$ when $k\to\infty$.

Proof. For small values of δ, we write $e^{A\delta}\approx I_d+\delta A$, $e^{-A\delta}\approx I_d-\delta A$, $e^{A\delta}x_k\approx x_k$. Then system (45) is equivalent to

$$\begin{aligned}\frac{x_{k+1}-x_k}{\delta}&=Ax_k-BB'P_k^{-1}(x_k-C'y_k),\\ P_{k+1}&=\lambda(I_d-\delta A)P_k(I_d-\delta A')+\delta BB'.\end{aligned}\tag{47}$$

Since the parameter $0<\lambda<1$, then it is possible to replace λ by $1-\beta$, where β is also a positive parameter which can be chosen in the interval $]0,1[$. This gives

$$\begin{aligned}P_{k+1}&=\lambda P_k-\delta\lambda P_kA'-\lambda\delta AP_k+\lambda\delta^2AP_kA'+\delta BB'\\ &=(1-\beta)P_k-\delta\lambda P_kA'-\lambda\delta AP_k\\ &\quad+\lambda\delta^2AP_kA'+\delta BB'.\end{aligned}\tag{48}$$

By neglecting the δ^2-power term, and dividing the two sides of the last equation by δ, we obtain

$$\frac{P_{k+1}-P_k}{\delta}=-\frac{\beta}{\delta}P_k-\lambda P_kA'-\lambda AP_k+BB'.\tag{49}$$

Passing to the limit $\delta\to 0$, and replacing x_k by $x(t)$, and P_k by $P(t)$, $\gamma=\beta/\delta$, we obtain exactly the continuous-time (14) if λ is close to 1. □

4.1. Discussion

Notice that the stability condition given by (46) comes from the standard stability result of the discrete-time Lypunov equation $P_{k+1}=\lambda e^{-A\delta}P_k e^{-A'\delta}+\delta BB'$. Choosing $\lambda<1$ prevents the matrix P_k from tending to zero and hence makes the discrete-time differentiator more alert to variations of the derivatives of y_k. The selection of the appropriate value of the parameter λ is a compromise between smoothing and closeness to the real derivatives of y_k. The basic degree of freedom in designing such differentiation systems is the choice of the differentiation order n. This property makes the construction of the discrete-time arbitrary-order differentiator straightforward, quite simple and easy to implement. For example, spline differentiation techniques (Craven & Wahba, 1979; Ibrir, 2000), and optimization-based differentiation algorithms suffer from heavy computational tool that involve and necessitate, in most cases, a complete redesign when the differentiation order change (Spriet & Bens, 1979).

To show the effectiveness of the discrete-time differentiator, consider again the observation problem of system (12). In real-time applications, the output is measured in discrete-time manner, so a lot of nonlinear observers are in need of the discretization of the dynamic model to conceive an observer. Here, we show that we can build an observer without discretizing the nonlinear model (12) since the unmeasured state $\xi_2(t)$ is given as a static function of $y(t)$ and

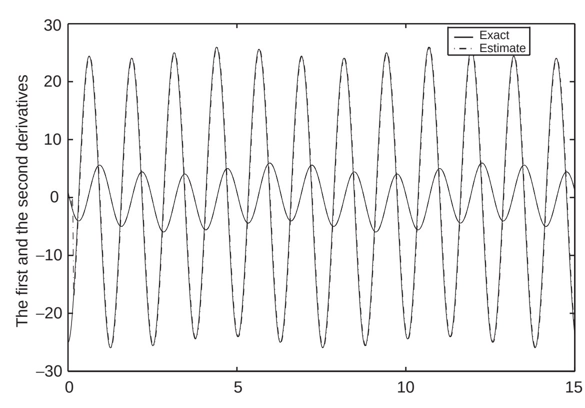

Fig. 1. The first derivatives and their estimates for $\gamma=200$, $H^{-1}(0)=0.01I_d$.

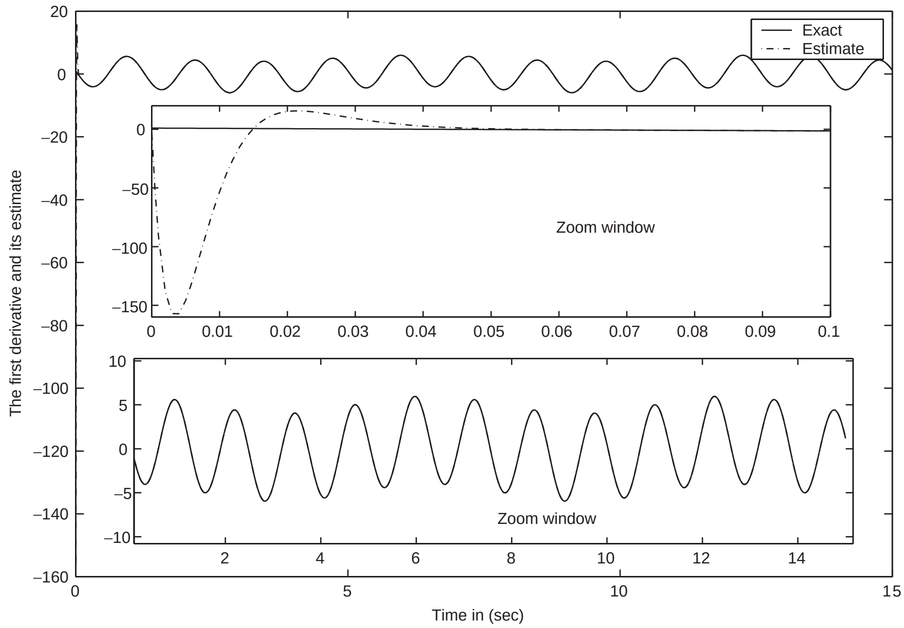

Fig. 2. The first derivative $\dot{y}$ and its estimate for $\varepsilon = \frac{1}{200}$, $k_i = C_3^i, 1 \leqslant i \leqslant 3$.

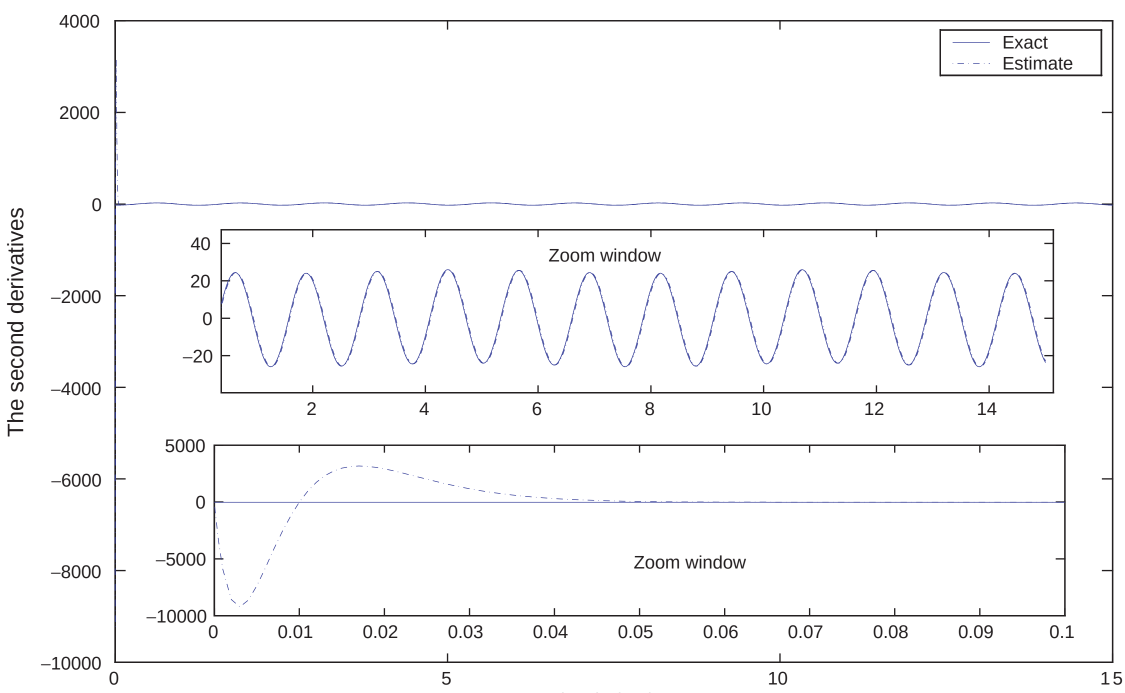

Fig. 3. The second derivative $\ddot{y}$ and its estimate for $\varepsilon = \frac{1}{200}$, $k_i = C_3^i, 1 \leqslant i \leqslant 3$.

$\dot{y}(t)$. The discrete-time observer is

$$
\begin{aligned}
x_{k+1} &= e^{A\delta}x_k - \delta BB'P_k^{-1}(e^{A\delta}x_k - C'y_k),\\
P_{k+1} &= \lambda e^{-A\delta}P_k e^{-A'\delta} + \delta BB',\\
\hat{\xi}_k(2) &= x_k(2) + y_k^3,
\end{aligned}
\tag{50}
$$

where $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 1}$, and $C \in \mathbb{R}^{1\times 2}$ are defined as in (15).

5. Illustrative examples

Let $y(t) = \sin(t) + \cos(5t)$ be a free analog signal which is supposed to be measured in continuous manner. In the

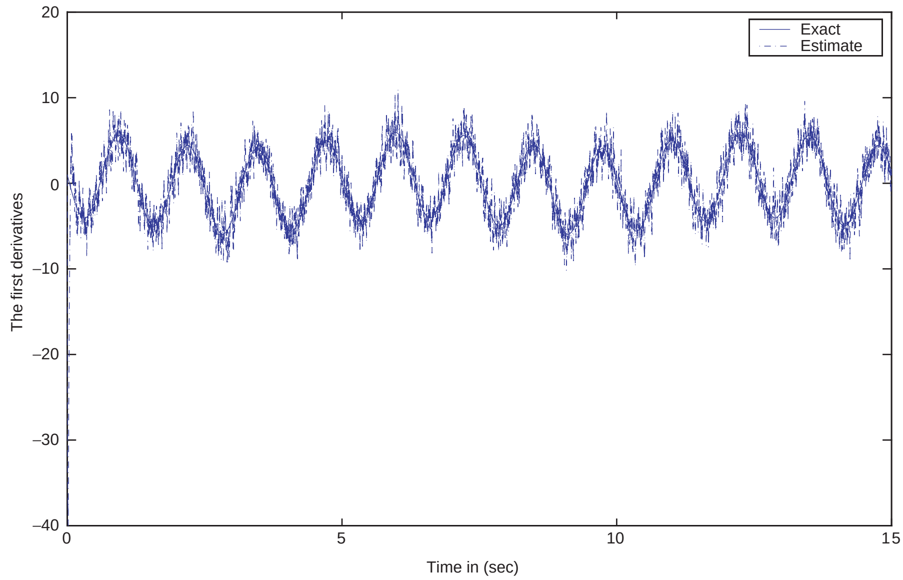

Fig. 4. The smooth exact derivative $\dot{y}$ and its estimate for $\varepsilon = \frac{1}{50}$, $n = 3$, $k_i = C_n^i, 1 \leqslant i \leqslant 3$.

following simulations, we compare the estimation qualities of differentiators (1) and (14) with those of the high-gain scheme

$$\dot{x}_i = x_{i+1} + k_i/\varepsilon^i(y - x_1), \qquad 1 \leqslant i \leqslant n-1$$

$$\dot{x}_n = k_n/\varepsilon^n(y - x_1), \tag{51}$$

proposed in Tornambè (1992) and Dabroom and Khalil (1999) in which ε is a small positive parameter and the polynomial $s^n + \sum_{i=1}^{n-1} k_i s^{n-i}$ is supposed to be Hurwitz. In Fig. 1, the estimates of the first and the second derivatives of the signal $y(t)$ are given by (1) for $\gamma = 200$ and $H^{-1}(0) = 0.01 I_{\mathrm{d}}$. In Figs. 2 and 3, we show that differentiator (51) exhibits important peaking while estimating the first derivatives of $y(t)$. For this simulation, we have taken $\varepsilon = \frac{1}{200}$ (i.e., $\gamma = 200$) and $k_i = C_3^i$ in order to compare it with the differentiation scheme (1) and maintain the same differentiation errors given by the two systems. By adding a white noise to the signal $y(t)$, we show that neither (1) nor (51) can provide satisfactory estimates of derivatives, see Fig. 4. In Fig. 5, we show that noise, differentiation error and peaking are more attenuated by the use of the differentiation scheme described in Section 3.2.2.

Indeed, the use of controllable canonical form differentiation observers offer a good compromise between the three contradictory performances: filtering, peaking, and error bound. Whereas, the observable canonical form-based differentiator (1) remains the better one if the peaking phenomenon should be avoided and the desired precision is high.

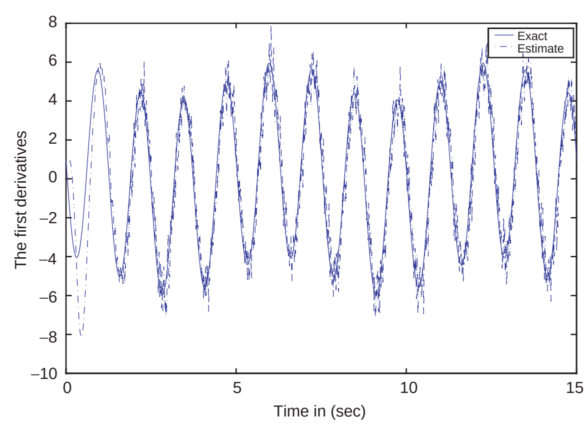

Fig. 5. The smooth exact derivative $\dot{y}$ and its estimate for $\gamma = 90$.

6. Conclusion

In this paper two new kinds of linear differentiation systems are discussed. The strength and weakness of each observer is outlined. It is showed that the compromise between the contradictory performances, expressed in terms of differentiation error, sensitivity to noise and peaking, seems to be tractable with controllable canonical form differentiators than with observable canonical form differentiation systems.

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