International Journal of Systems Science
Vol. 37, No. 15, 15 December 2006, 1097–1109

On observer design for nonlinear systems

S. IBRIR*

Department of Mechanical & Industrial Engineering, Concordia University,
1455 de Maisonneuve Blvd. West, Montreal, Quebec, H3G 1M8, Canada

(*Received 17 September 2002; in final form 29 June 2006*)

The main weakness of all control methodologies is the dependency of feedbacks to full state measurements. In practical situations, measuring the states of a given system may fail because sometimes the measurements are impossible and sometimes, possible, but too expensive. Observer design for highly nonlinear dynamics is an important issue, particularly when the locally observable dynamics are not linearly observable. In such circumstances the ability to reduce the system to observable or observer form is key to observer design. This paper provides two observers for nonlinear systems given in Brunovski form. The first observer is a high-gain observer with a classical output injection form, while the second is a high-gain observer with a q-integral path. Finally, the discrete-time implementation of the high-gain observer is discussed in linear matrix inequality framework. A motivating example is shown to highlight the efficacy of the developed observers.

Keywords: Nonlinear observers; System theory; Linear matrix inequalities; Discretization; Filtering; Numerical differentiation

1. Introduction

Nonlinear observers are a central part of control engineering, estimation and fault detection as well as regulator approaches to reconfigurable control systems. It is known that the control of dynamical systems is often based on state feedbacks to achieve desired properties of the closed loop system. Unfortunately, in many applications, the exact state of the system is not available online. The problem of estimating the state of a dynamical system from outputs and inputs (commonly known as observing the state, hence the name observer) is a crucial problem in the theory of systems. For linear systems, it has been extensively studied, and has proven extremely useful, especially for control applications such as observer-based-control design. However, for nonlinear systems, the theory of observers is not nearly as complete nor successful as it is for linear systems.

When the dynamics of a system involves nonlinearities, issues of observability and observer design present new difficulties or complexities that are absent in linear problems. For example, in linear systems, the input does not play a role in deciding observability but a nonlinear system may be observable for some inputs and not so for others. As a result, new theoretical paradigms for observer design for nonlinear systems have emerged over the past decade. The problem of estimating the states of a dynamical system from partial measurements has a long history. The extended Kalman filter (Kalman 1960, Zeitz 1987b, Song and Grizzle 1992, Reif and Unberhauen 1999, Reif *et al*. 1999) is one of the widely used alternative methods for estimating the states of a nonlinear system. It is obtained by linearizing the dynamics and the observation along the trajectory of the estimate. However, this is only a local method, in the sense that the estimate converges to the true state if the initial error is not too large and the linearization does not present any singularity. Further results on observer design based on error linearization, Lyapunov techniques, linearization by input-output injection, and numerical techniques are extensively discussed in the references (Li and Tao 1986, Xia and Gao 1988, Yaz 1988, Tsinias 1989, 1990a, 1990b, Ding *et al*. 1990, Phelps 1991, Gauthier *et al*. 1992,

*Email: sibrir@encs.concorida.ca

ISSN 0020–7721 print/ISSN 1464–5319 online © 2006 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/00207720601014081

Tornambé 1992, Deza *et al.* 1993, Proychev and Mishkov 1993, Glumineau and Lopez-Morales 1999, V. Lopez-Morales 1999, Ibrir 1999, 2001, Arcak and Kokotović 2001).

Note that the literature contains numerous design strategies for systems that are linearly observable. When this is not the case, available techniques are far more limited. Moreover, application experience from which to draw conclusions about their relative practical merits is virtually non-existent. One reason for this is, undoubtedly, the lack of computational tools. The ability to reduce the system to observable or observer form is fundamental to nonlinear observer design, and it is the main focus of this paper.

High-gain observers are quite popular in system theory and a huge part of the literature has been devoted to construction of such observers, see e.g., Thau (1973), Kou *et al.* (1975), Krener and Isidori (1983), Krener and Respondek (1985), Slotine *et al.* (1987), Zeitz, (1987a), Xia and Gao (1989), Misawa and Hedrick (1989), Phelps (1991), Tornambé (1992), Gauthier *et al.* (1992), Ciccarella *et al.* (1993a, b) Raghavan and Hedrick (1994), Kazantzis and Kravaris (1998), Rajamani (1998), Reif *et al.* (1998). The reader is also referred to some new contributions in observer design Guay (2002), Krener and Xiao (2002a, b), Kreisselmeier and Engel (2003), Ibrir (2003, 2004). In the discrete-time case the contributions are also numerous, see for example Lee and Nam (1991a, b), Ciccarella *et al.* (1993b), Moraal and Grizzle (1995), Reif and Unberhauen (1999), and Reif *et al.* (1999). Although high-gain observers provide certain robustness against unmodelled dynamics, the main weaknesses of this kind of state estimators is noise amplification through high-gain observer gains. Therefore, filtering of the estimates remains one of the major issues that necessitates, in most cases, a complete redesign of the observer gain.

In this paper, we continue our investigations on high-gain observer design. We concentrate particularly on observation of either single output nonlinear systems that appear naturally in Brunovski form or systems that can be transformed into this form under the uniform observability condition. In our design, we associate to the observer dynamics a parameter-dependent Riccati equation that defines the solution of the observer gain for a given Lipschitz constant. According to this formulation, the stability of the observation error along with state filtering are easily elaborated. Conceptually, the technique used herein is the same as that proposed in Ciccarella *et al.* (1993b), in the sense that the poles of the linear error dynamics can be freely assigned. However, our observer formulation, in a Kalman setting, gives additional information about the optimality of the observer. Since the dynamics of the nonlinear system can be rewritten as a linear system subject to a norm-bounded perturbation, it will be shown that our high-gain observer behaves as a robust deterministic Kalman observer for Lipschitzian nonlinear systems. Subsequently, we show how to make the proposed high-gain observer robust against measurement errors, generally encountered in practice. One of the main contribution of this paper is to propose a q-integral nonlinear observer that handles the effect of noise and gives a solution to both state filtering and stability of the observation error dynamics. It is proved that the output uncertainty is enfeebled by increasing the value of the output integral order q. In contrast to general high gains observers with classical proportional injection terms, the proposed robust differentiation observer offers noise reduction property with a prescribed degree of stability. This is done by injecting the qth path of the measured outputs instead of the usual noisy outputs. Illustrative example clarifying this fact will be included with some numerical simulations.

Since digital implementation of high-gain observers presents some difficulties, the second part of this paper will be devoted to the discrete-time implementation of the developed high-gain observer and how to chose properly the sampling period such that the states of the discretized observer converge asymptotically to the discrete system states. In this part, we highlight the connection between constructing a discrete observer for the Euler discrete scheme of the nonlinear system and the Euler discretization of a continuous-time nonlinear observer. The breakdown of the developed discrete-time observers is given in linear matrix inequality framework.

The paper is organized as follows. Section 2 contains two main subsections. The first one concerns the theory of the nonlinear observer and the second is entirely devoted to the q-integral nonlinear observer. In section 3, the discrete-time implementation of the developed continuous-time observers is discussed. In section 4, an illustrative example is provided to show the effectiveness of such observation strategy. Finally, we end with some concluding remarks.

Preliminaries and notations

- $\mathbb{R}$ is the set of real numbers. $\mathbb{Z}_{\geq 0}$ stands for the set of positive integer numbers.
- $\|\cdot\|$ denotes the usual Euclidean norm.
- If A and B are two real matrices, then $A > B$ is equivalent to $A - B$ positive definite.
- A' denotes the matrix transpose of A.

- $\boldsymbol{I}$ is the identity matrix of appropriate dimensions.
- $\mathbf{0}$ is the null matrix of appropriate dimensions.
- $\lambda_{\min}(A)$ is the smallest eigenvalue of the matrix A, and $\lambda_{\max}(A)$ stands for the largest eigenvalue of A.
- $C_n^k = n!/k!(n-k)!$ is the binomial coefficient.
- SISO: Single-input Single-output.
- the star "$\star$" symbol is used to show an element induced by transposition.

For the clarity of the statement proofs, we would rather present some basic lemmas.

Lemma 1 (The schur complement lemma) (Boyd *et al.* 1994)**:** *Given constant matrices M, N, Q of appropriate dimensions where M and Q are symmetric, then $Q > 0$ and $M + N'Q^{-1}N < 0$ if and only if*

$$\begin{bmatrix} M & N' \\ N & -Q \end{bmatrix} < 0,$$

or equivalently

$$\begin{bmatrix} -Q & N \\ N' & M \end{bmatrix} < 0.$$

Lemma 2: *For any constant symmetric matrix $M \in \mathbb{R}^{n\times n}$, $M = M' > 0$, scalar $\gamma > 0$, vector function $\omega: [0, \gamma] \mapsto \mathbb{R}^n$ such that the integration in the following is well defined, we have*

$$\gamma \int_0^\gamma \omega'(\beta) M \omega(\beta) \mathrm{d}\beta \geq \left(\int_0^\gamma \omega(\beta)\mathrm{d}\beta \right)' M \left(\int_0^\gamma \omega(\beta)\mathrm{d}\beta \right). \tag{1}$$

Proof: See Gu (2000).

2. Observer design

2.1. *ARE-based high-gain observer*

A commonly used model for a broad class of physical phenomena is the nonlinear input–output differential equation

$$y^{(n)}(t) = \phi\big(y, \dot{y}, \ddot{y}, \ldots, y^{(n-1)}, u\big)(t), \tag{2}$$

where $x(t) = (y \quad \dot{y} \quad \cdots \quad y^{(n-1)})'(t) \in \mathscr{M} \subset \mathbb{R}^n$ (a neighborhood of $x_0 \in \mathbb{R}^n$), $u \in \mathscr{U}$ is m-vector and $\mathscr{U}$ is the set of bounded inputs that makes system (2) observable, and $y(t) \in \mathbb{R}$. We assume that x_0 is an equilibrium point corresponding to zero input and output, i.e., $\phi(x_0) = 0$. The function $\phi(\cdot)$ is supposed to be smooth. For notation simplicity time t is omitted from the state space representations, and $\dot{x}$ stands for the differentiation of $x(t)$ with respect to time. System (2) admits the state-space representation

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ &\vdots \\ \dot{x}_i &= x_i + 1, \\ &\vdots \\ \dot{x}_n &= \phi(x, u), \\ y &= x_1. \end{aligned}\right\} \tag{3}$$

Constructing the unmeasured states via high-gain observers is an old problem (Thau 1973, Kou *et al.* 1975). The first investigations return to the work of Thau (1973) in which a straightforward approach to observer design is presented. Overcoming nonlinearities with the use of high-gain linear feedback seems to be very useful, but the available techniques do not furnish clear insights into properly choosing the observer gain. Recently, Rajamani (1998) proposed sufficient conditions for the existence of the observer gain that ensures the decay of the observation error. Raghavan and Hedrick (1994) proposed a design algorithm for choosing the feedback (observer gain) that guarantees the stability of the observer error. The developed algorithm depends on the solvability of an algebraic Riccati equation that depends on the Lipschitz constant of the nonlinearity and a design parameter ϵ. In this section, we propose a similar type of ARE-based high-gain observer whose states converge asymptotically to the exact ones with arbitrary rate of convergence. In our design, the solution of the algebraic Riccati equation always exists and offers more freedom to choose the poles of the closed loop of the error dynamics. The design strategy is given in the following theorem.

Theorem 1: *Consider the SISO nonlinear system* (2) *where $\phi(x,u)$ is supposed to be globally Lipschitz, e.g.; for any $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$*

$$\|\phi(x_1, u) - \phi(x_2, u)\| \leq \rho \|x_1 - x_2\|. \tag{4}$$

If γ is chosen such that the condition

$$\rho \leq \frac{\lambda_{\min}\big(P^{-1/2}(\gamma) Q(\gamma) P^{-1/2}(\gamma)\big)}{2\sqrt{\bar{p}_{n,n}}\lambda_{\max}(P(\gamma))} \tag{5}$$

holds, then the following system

$$\left.\begin{aligned} \dot{\hat{x}} &= A\hat{x} + f(\hat{x}, u) + P(\gamma)C'(y - C\hat{x}), \\ AP(\gamma) &+ P(\gamma)A' - P(\gamma)C'CP(\gamma) + Q(\gamma) = 0 \end{aligned}\right\} \tag{6}$$

is an exponential observer for system (3). *The nominal matrices of the nonlinear observer are*

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}' \in \mathbb{R}^n, \; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n \tag{7}$$

and

$$\left.\begin{aligned} f(x,u) &= B\phi(x,u), \\ Q(\gamma) &= \mathrm{diag}\big[q_{1,1}(\gamma),\ldots,q_{n,n}(\gamma)\big], \\ q_{1,1}(\gamma) &= \big(\alpha_1^2 - 2\alpha_2\alpha_0\big)\gamma^2, \\ q_{i,i}(\gamma) &= \left(\alpha_i^2 + 2\sum_{k=i}^{n-1}(-1)^{k+i-1}\alpha_{k+1}\alpha_{k-1}\right)\gamma^{2i}, \\ &\quad (2 \le i \le n-1), \\ q_{n,n}(\gamma) &= \alpha_n^2\gamma^{2n}, \end{aligned}\right\} \tag{8}$$

where $\alpha_0 = 1$, $\alpha_k = 0$ *for* $0 > k \ge n+1$, *and the reals* $(1 \le \alpha_k \le n)$ *must be selected such that*

$$s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n = 0 \tag{9}$$

is Hurwitz and

$$\alpha_i^2 + 2\sum_{k=i}^{n-1}(-1)^{k+i-1}\alpha_{k+1}\alpha_{k-1} > 0. \tag{10}$$

We note $\bar{p}_{n,n} = \big(P^{-1}\big)_{n,n}$, *and* $P^{1/2}(\gamma)$ *stands for the square root matrix of* $P(\gamma)$.

Before giving the complete proof of the last theorem, we need to introduce the following lemmas.

Lemma 3: *Let* $P_1(\gamma_1)$ *and* $P_2(\gamma_2)$ *be two symmetric, positive definite matrices, solutions of the algebraic Riccati equations*

$$\left.\begin{aligned} AP_1(\gamma_1) + P_1(\gamma_1)A' - P_1(\gamma_1)C'CP_1(\gamma_1) + Q(\gamma_1) = 0, \\ AP_2(\gamma_2) + P_2(\gamma_2)A' - P_2(\gamma_2)C'CP_2(\gamma_2) + Q(\gamma_2) = 0, \end{aligned}\right\} \tag{11}$$

for certain set of constants $(\alpha_i)_{1\le i\le n}$ *satisfying condition* (10). *Then for any positive constants* γ_1 *and* γ_2 *such that* $\gamma_1 > \gamma_2$, *we have* $P_1(\gamma_1) > P_2(\gamma_2)$.

Proof: The existence of positive definite solutions $P_1(\gamma)$ and $P_2(\gamma)$ of the algebraic Riccati equations (11) is guaranteed by the observability condition of the pair (A, C) and the positive definiteness property of $Q(\gamma_1)$ and $Q(\gamma_2)$ issued from condition (10). From (11), we form the difference $P_1(\gamma_1) - P_2(\gamma_2)$, we obtain

$$\begin{aligned} &A(P_1(\gamma_1) - P_2(\gamma_2)) \\ &\quad + (P_1(\gamma_1) - P_2(\gamma_2))A' - P_1(\gamma_1)C'CP_1(\gamma_1) \\ &\quad + P_2(\gamma_2)C'CP_2(\gamma_2) + Q_1(\gamma_1) - Q_2(\gamma_2) = 0. \end{aligned} \tag{12}$$

The last equation is rewritten as

$$\begin{aligned} &(A - P_2(\gamma_2)C'C)(P_1(\gamma_1) - P_2(\gamma_2)) \\ &\quad + (P_1(\gamma_1) - P_2(\gamma_2))(A - P_2(\gamma_2)C'C)' \\ &\quad - (P_1(\gamma_1) - P_2(\gamma_2))C'C(P_1(\gamma_1) - P_2(\gamma_2)) \\ &\quad + Q_1(\gamma_1) - Q_2(\gamma_2) = 0. \end{aligned} \tag{13}$$

If we put $X(\gamma_1,\gamma_2) = P_1(\gamma_1) - P_2(\gamma_2)$, $\Xi = A - P_2(\gamma_2)C'C$, $Q(\gamma_1,\gamma_2) = Q_1(\gamma_1) - Q_2(\gamma_2)$, equation (13) is exactly the algebraic Riccati equation

$$\begin{aligned} &\Xi X(\gamma_1,\gamma_2) + X(\gamma_1,\gamma_2)\Xi' \\ &\quad - X(\gamma_1,\gamma_2)C'C'X(\gamma_1,\gamma_2) + Q(\gamma_1,\gamma_2) = 0. \end{aligned} \tag{14}$$

Since the pair $(A - P_2(\gamma_2)C'C, C)$ is observable, then (14) admits a positive definite solution $X(\gamma_1,\gamma_2) > 0$, which implies that $P_1(\gamma_1) > P_2(\gamma_2)$. To prove the main statement of this section, we need to introduce the following lemma.

Lemma 4: *If* A, C *and* $Q(\gamma)$ *are defined as in Theorem* 1, *then the observer gain is given by*

$$P(\gamma)C' = G'(\gamma) = \begin{bmatrix} \alpha_1\gamma \\ \alpha_2\gamma^2 \\ \vdots \\ \alpha_n\gamma^n \end{bmatrix}, \tag{15}$$

where $P(\gamma)$ *is the solution of the ARE equation*

$$AP(\gamma) + P(\gamma)A' - P(\gamma)C'CP(\gamma) + Q(\gamma) = 0.$$

Proof: Let

$$P(\gamma) = \begin{bmatrix} p_1 & p_2 & p_3 & \cdots & p_n \\ p_2 & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ p_3 & p_{n+2} & p_{2n} & \cdots & p_{3n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_n & p_{2n-1} & p_{3n-3} & \cdots & p_{n^2+n/2} \end{bmatrix}(\gamma). \quad (16)$$

be the solution of the ARE equation

$$AP(\gamma) + P(\gamma)A' - P(\gamma)C'CP(\gamma) + \operatorname{diag}\left[q_{1,1}(\gamma), q_{2,2}(\gamma), \cdots, q_{n,n}(\gamma)\right] = 0.$$

The n algebraic equations of the variables $p_1(\gamma)$, $p_2(\gamma), \ldots, p_n(\gamma)$ are written as follows

$$\begin{aligned} &2p_2(\gamma) - p_1^2(\gamma) + q_{1,1}(\gamma) = 0,\\ &2p_1(\gamma)p_3(\gamma) - p_2^2(\gamma) + q_{2,2}(\gamma) = 0,\\ &\vdots\\ &\cdots - 2p_{i+2}(\gamma)p_{i-2}(\gamma) + 2p_{i+1}(\gamma)p_{i-1}(\gamma)\\ &\quad - p_{i^2}(\gamma) + q_{i,i}(\gamma) = 0,\\ &\vdots\\ &2p_n(\gamma)p_{n-2}(\gamma) - p_{n-1}^2(\gamma) + q_{n-1,n-1}(\gamma) = 0,\\ &-p_n^2(\gamma) + q_{n,n} = 0, \end{aligned}$$

where

$$\left.\begin{aligned} &q_{1,1}(\gamma) = \left(\alpha_1^2 - 2\alpha_2\alpha_0\right)\gamma^2,\\ &q_{i,i}(\gamma) = \left(\alpha_i^2 - 2\alpha_{i+1}\alpha_{i-1} + 2\alpha_{i+2}\alpha_{i-2} - \cdots\right)\gamma^{2i},\\ &\qquad (2 \le i \le n-1),\\ &q_{n,n}(\gamma) = \alpha_n^2\gamma^{2n}. \end{aligned}\right\} \quad (17)$$

This immediately gives

$$p_i(\gamma) = \alpha_i\,\gamma^i, 1 \le i \le n. \quad (18)$$

The remaining elements of $P(\gamma)$ can be obtained from the solution of the Lyapunov matrix equation

$$AP(\gamma) + P(\gamma)A' = G'(\gamma)G(\gamma) - Q(\gamma). \quad (19)$$

The solution of (19) is

$$(P)_{i,j}(\gamma) = \widetilde{P}_{i,j}\,\gamma^{i+j-1}, \quad (20)$$

where $\widetilde{P}$ is the solution of the Lyapunov equation

$$A\widetilde{P} + \widetilde{P}A' = G'(\gamma)G(\gamma)\big|_{\gamma=1} - Q(\gamma)\big|_{\gamma=1}. \quad (21)$$

Now, we are ready to give the proof of Theorem 1.

Proof of theorem 1: *Put $e = x - \hat{x}$, then the observer error verifies the dynamic equation*

$$\dot{e} = (A - P(\gamma)C'C)e + f(x, u) - f(\hat{x}, u). \quad (22)$$

With the Lyapunov function $V(e) = e'P^{-1}e$, we have

$$\begin{aligned} \dot{V}(e) &= \dot{e}'P^{-1}(\gamma)e + e'P^{-1}(\gamma)\dot{e}\\ &= e'\left(A'P^{-1}(\gamma) + P^{-1}(\gamma)A - 2C'C\right)e\\ &\quad + 2(f(x,u) - f(\hat{x},u))'P^{-1}(\gamma)e. \end{aligned} \quad (23)$$

Using the second equation of system (6), we obtain

$$\begin{aligned} \dot{V}(e) &= e'\left(-P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma) - C'C\right)e + 2(f(x,u)\\ &\quad -f(\hat{x},u))'P^{-1}(\gamma)e\\ &\le e'\left(-P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma)\right)e + 2(f(x,u) - f(\hat{x},u))'P^{-1}e\\ &\le -\lambda_{\min}\left(P^{-1/2}(\gamma)Q(\gamma)P^{-1/2}(\gamma)\right)\left\|P^{-1/2}(\gamma)e\right\|^2\\ &\quad + 2\left\|(f(x,u) - f(\hat{x},u))'P^{-1/2}(\gamma)\right\|\left\|P^{-1/2}e\right\|\\ &= -\lambda_{\min}\left(P^{-1/2}(\gamma)Q(\gamma)P^{-1/2}(\gamma)\right)\left\|P^{-1/2}(\gamma)e\right\|^2\\ &\quad + 2\sqrt{(f(x,u) - f(\hat{x},u))'P^{-1}(\gamma)(f(x,u) - f(\hat{x},u))}\\ &\quad \left\|P^{-1/2}(\gamma)e\right\|\\ &= -\lambda_{\min}\left(P^{-1/2}(\gamma)Q(\gamma)P^{-1/2}(\gamma)\right)\left\|P^{-1/2}(\gamma)e\right\|^2\\ &\quad + 2\sqrt{\bar{p}_{n,n}}\left\|f(x,u) - f(\hat{x},u)\right\|\left\|P^{-1/2}(\gamma)e\right\|\\ &\le -\lambda_{\min}\left(P^{-1/2}(\gamma)Q(\gamma)P^{-1/2}(\gamma)\right)\left\|P^{-1/2}(\gamma)e\right\|^2\\ &\quad + 2\rho\sqrt{\bar{p}_{n,n}}\|e\|\left\|P^{-1/2}(\gamma)e\right\|\\ &\le -\lambda_{\min}\left(P^{-1/2}(\gamma)Q(\gamma)P^{-1/2}(\gamma)\right)\left\|P^{-1/2}(\gamma)e\right\|^2\\ &\quad + 2\rho\sqrt{\bar{p}_{n,n}}\left\|P^{1/2}\right\|\left\|P^{-1/2}e\right\|^2. \end{aligned}$$

Finally, we have

$$\begin{aligned} \dot{V}(e) \le -\Bigg(&\lambda_{\min}\left(P^{-1/2}(\gamma)Q(\gamma)P^{-1/2}(\gamma)\right)\\ &- 2\rho\sqrt{\bar{p}_{n,n}\lambda_{\max}(P(\gamma))}\Bigg)\left\|P^{-1/2}(\gamma)e\right\|^2. \end{aligned} \quad (24)$$

We conclude that if $\lambda_{\min}\left(P^{-1/2}(\gamma)Q(\gamma)P^{-1/2}(\gamma)\right) > 2\rho\sqrt{\bar{p}_{n,n}\lambda_{\max}(P(\gamma))}$, then $\dot{V}$ is always negative and consequently, the observer error decays exponentially to zero.

Now we shall prove that the last condition can be always verified. Using result of Lemma (4), we have

$$\bar{p}_{n,n} = \frac{\left(\widetilde{P}\right)^{-1}_{n,n}}{\gamma^{2n-1}}. \tag{25}$$

In addition

$$\sqrt{\lambda_{\max}(P(\gamma))} = \left\| P^{1/2}(\gamma) \right\| \leq \sqrt{n} \left\| P^{1/2}(\gamma) \right\|_{\infty}. \tag{26}$$

Since the largest value of $P^{1/2}(\gamma)$ is proportional to $\sqrt{\gamma^{2n-1}}$, then $\left(\lambda_{\max}(P(\gamma))\bar{p}_{n,n}\right)^{1/2}$ is a rational function of γ and the dimension n. So when γ increases the quantity $\sqrt{\lambda_{\max}(P(\gamma))\bar{p}_{n,n}}$ remains constant. In the other hand, since $P_{i,j} = \widetilde{P}_{i,j}\gamma^{i+j-1}$ and $Q_{i,i}(\gamma) = \beta_i \gamma^{2i}$ then, we conclude that we can always find some constants $c_{i,i}$ such that

$$P_{i,i}(\gamma) = \frac{c_{i,i}}{\gamma} Q_{i,i}(\gamma). \tag{27}$$

This implies that we can find a positive number λ_n which depends on n such that

$$Q(\gamma) - \frac{\gamma}{\lambda_n} P(\gamma) > 0. \tag{28}$$

Then using the last inequality, we have for any $\gamma_1 > \gamma_2$

$$\left.\begin{array}{l} P_1^{-1/2}(\gamma_1)Q(\gamma_1)P_1^{-1/2}(\gamma_1) > \dfrac{\gamma_1}{\lambda_n}\boldsymbol{I}, \\ P_2^{-1/2}(\gamma_2)Q(\gamma_2)P_2^{-1/2}(\gamma_2) > \dfrac{\gamma_2}{\lambda_n}\boldsymbol{I}, \end{array}\right\} \tag{29}$$

where $P_1(\gamma)$ and $P_2(\gamma)$ are defined as in Lemma 3. This gives

$$P_1^{-1/2}(\gamma_1)Q(\gamma_1)P_1^{-1/2}(\gamma_1) - P_2^{-1/2}(\gamma_2)Q(\gamma_2)P_2^{-1/2}(\gamma_2) > \left(\frac{\gamma_1}{\lambda_n} - \frac{\gamma_2}{\lambda_n}\right)\boldsymbol{I} > 0. \tag{30}$$

Consequently,

$$\lambda_{\min}\left(P_1^{-1/2}(\gamma_1)Q(\gamma_1)P_1^{-1/2}(\gamma_1)\right) > \lambda_{\min}\left(P_2^{-1/2}(\gamma_2)Q(\gamma_2)P_2^{-1/2}(\gamma_2)\right). \tag{31}$$

Finally, we conclude that for any constant ρ, we can find $\gamma > 0$ such that inequality (5) holds.

From result of Lemma 4, we see that the poles of the linear part of the observer error dynamics can be placed arbitrarily in the left half plan according to the specific choice of the Hurwitz polynomial $s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n = 0$. When $\alpha_k = C_n^k$ the observer (6) is exactly the same high-gain observer proposed by Gauthier *et al* (1992). Conceptually, our technique is the same as the one proposed in Ciccarella *et al*. (1993b). However, the proposed formulation gives additional information about the optimality of the observer. This can be seen from the fact that observer (6) is merely a robust deterministic Kalman observer for the following system

$$\dot{x} = \left(A + \Delta A(x,u)\right)x, \tag{32}$$

where $\Delta A(x,u) = B\int_0^1 \partial f(s,u)/\partial s|_{s=\lambda x}\, \mathrm{d}\lambda$ is an n by n bounded matrix for bounded control input $u \in \mathcal{U}$. The norm of the perturbation term $\Delta A(x,u)$ depends essentially on the applied control input u and the form of the slopes of nonlinearities. By analogy with the linear time-invariant case, we show that observer (6) minimizes a quadratic cost function that depends on the weighting matrices of the ARE. The optimality of such an observer is given in the following statement.

Corollary 1: *For a given γ satisfying the condition of Theorem* 1, *there exists $\beta > 0$, that depends on γ and the Lipschitz constant ρ, such that the following integral inequality constraint*

$$\int_0^t \beta e'(\tau)Q^{-1}(\gamma)e(\tau) + \left(C\hat{x}(\tau) - y(\tau)\right)'\left(C\hat{x}(\tau) - y(\tau)\right)\mathrm{d}\tau \leq e'(0)P^{-1}(\gamma)e(0) \tag{33}$$

is verified along the trajectories of observer (6).

Proof: For any $t > 0$, we have

$$\int_0^t \left(C\hat{x}(\tau) - y(\tau)\right)'\left(C\hat{x}(\tau) - y(\tau)\right)\mathrm{d}\tau \leq \int_0^t e'(\tau)C'Ce(\tau)\mathrm{d}\tau + V(e(t)), \tag{34}$$

where $V(e(t)) = e'(t)P^{-1}(\gamma)e(t)$. This implies that

$$\begin{aligned} \int_0^t & \left(C\hat{x}(\tau) - y(\tau)\right)'\left(C\hat{x}(\tau) - y(\tau)\right)\mathrm{d}\tau \\ & \leq \int_0^t \left[e'(\tau)C'Ce(\tau) + \dot{V}(e(t)) \right]\mathrm{d}\tau + V(e(0)) \\ & = \int_0^t -e'(\tau)\left[P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma) \right]e'(\tau)\mathrm{d}\tau \\ & + \int_0^t 2e'(\tau)P^{-1}(\gamma)\left[f(x(\tau),u(\tau)) - f(\hat{x}(\tau),u(\tau)) \right]\mathrm{d}\tau + V(e(0)). \end{aligned} \tag{35}$$

Let $c_1 = \lambda_{\min}\left(P^{-1/2}(\gamma)Q(\gamma)P^{-1/2}(\gamma)\right) - 2\rho\sqrt{\bar{p}_{n,n}\lambda_{\max}(P(\gamma))}$. From (27), we can always find

$\varepsilon > 0$ that depends on the dimension n such that $P(\gamma) < \varepsilon/\gamma Q(\gamma)$. This gives $-P^{-1}(\gamma) < -(\gamma/\varepsilon)Q^{-1}(\gamma)$. If we put $\beta = (c_1\gamma/\varepsilon)$, then for any γ satisfying condition of Theorem 1, and by the use of (24), we can write

$$\begin{aligned}\int_0^t \left(C\hat{x}(\tau) - y(\tau)\right)'\left(C\hat{x}(\tau) - y(\tau)\right)\mathrm{d}\tau &\le \int_0^t -c_1 e'(\tau)P^{-1}(\gamma)e(\tau)\mathrm{d}\tau \\ &\quad + e'(0)P^{-1}(\gamma)e(0), \\ &\le \int_0^t -\beta e'(\tau)Q^{-1}(\gamma)e(\tau)\mathrm{d}\tau \\ &\quad + e'(0)P^{-1}(\gamma)e(0), \end{aligned} \tag{36}$$

which is the claim.

2.2. *q-Integral nonlinear observer*

In our previous observer scheme, the high-gain output injection is conceived to defeat the inherent nonlinearities. However, this proportional injection arises noise amplification through the high-gain output injection term. This serious drawback, that is generally encountered in such observation schemes, makes the filtering of the estimates almost impossible when the system nonlinearity is of a large Lipschitz constant. In this subsection, we plan to reformulate the high-gain observation scheme by replacing the proportional P injection term with a multiple integral injection term that involves the qth integral of the output. Actually, the notion of adding an integral path is not quite new. The first idea of proportional integral PI observers has been proposed by Wojciechwski (1978) and further developed by Beale and Shafai (1989) and Niemann *et al.* (1995).

The aim of this subsection is to use the result of the last subsection to build another observer that behaves more resistant to measurement errors of high levels. The basic idea is to augment first the original system with q integrators and feed back the observer dynamics with the exact qth integral of the noisy output. The amount of noise that may contain the system output will be enfeebled with the presence of the successive q integrators. Consider the system

$$\left.\begin{aligned} \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= \xi_3, \\ &\vdots \\ \dot{\xi}_q &= y, \\ \dot{x}_1 &= x_2, \\ &\vdots \\ \dot{x}_n &= \phi(x,u), \\ \tilde{y} &= \xi_1, \end{aligned}\right\} \tag{37}$$

where $y = x_1 + w$ is the noisy output of system (2), and $w = w(t)$ is a time-dependent norm-bounded noise of high frequency. In matrix notation, system (37) is rewritten as

$$\left.\begin{aligned} \dot{\eta} &= \widetilde{A}\eta + Bw + \widetilde{f}(\eta,u), \\ \widetilde{y} &= \widetilde{C}\eta, \end{aligned}\right\} \tag{38}$$

where

$$\left.\begin{aligned} &\eta = \begin{bmatrix} \xi \\ x \end{bmatrix}, \quad \widetilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(n+q)\times(n+q)}, \\ &B = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n+q}, \quad \widetilde{f}(\eta,u) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \phi(x,u) \end{bmatrix} \in \mathbb{R}^{n+q}, \\ &\widetilde{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n+q}. \end{aligned}\right\} \tag{39}$$

When $w \equiv 0$, system (38) is in form of system (2), and then observer (6) can be applied. The observer is readily constructed as follows

$$\begin{aligned} \dot{\hat{\eta}} &= \widetilde{A}\hat{\eta} + \widetilde{f}(\hat{\eta},u) + Z(\gamma)\widetilde{C}'\left(\widetilde{y} - \widetilde{C}\hat{\eta}\right), \\ &\widetilde{A}Z(\gamma) + Z(\gamma)\widetilde{A}' - Z(\gamma)\widetilde{C}'CZ(\gamma) + \widetilde{Q}(\gamma) = 0, \end{aligned} \tag{40}$$

where $\widetilde{Q}(\gamma) \in \mathbb{R}^{(n+q)\times(n+q)}$ are defined as in subsection 2.1. Let $e = \hat{\eta} - \eta$ be the observation error, then we have

$$\dot{e} = \left(\widetilde{A} - Z(\gamma)\widetilde{C}'\widetilde{C}\right)e + \widetilde{f}(\hat{\eta},u) - \widetilde{f}(\eta,u) - Bw. \tag{41}$$

The derivative of the Lyapunov function $V(e) = e'Z^{-1}(\gamma)e$ along the trajectories of system (41) is

$$\begin{aligned} \dot{V}(e) &= e'\left(\widetilde{A}'Z^{-1}(\gamma) + Z^{-1}(\gamma)\widetilde{A} - 2\widetilde{C}'\widetilde{C}\right)e \\ &\quad + 2e'Z^{-1}(\gamma)\left(\widetilde{f}(\hat{\eta},u) - \widetilde{f}(\eta,u)\right) \\ &\quad - 2e'Z^{-1}(\gamma)Bw. \end{aligned} \tag{42}$$

From (40), we have

$$\widetilde{A}'Z^{-1}(\gamma) + Z^{-1}(\gamma)\widetilde{A} + Z^{-1}(\gamma)\widetilde{Q}(\gamma)Z^{-1}(\gamma) - \widetilde{C}'\widetilde{C} = 0. \tag{43}$$

This gives

$$\begin{aligned}\dot{V}(e) &\le -e'\Big(Z^{-1}(\gamma)\widetilde{Q}(\gamma)Z^{-1}(\gamma)\Big)e \\ &\quad + 2e'Z^{-1}(\gamma)\Big(\widetilde{f}(\hat{\eta},u) - \widetilde{f}(\eta,u)\Big) \\ &\quad - 2e'Z^{-1}(\gamma)Bw \\ &\le -\Bigg(\lambda_{\min}\Big(Z^{-1/2}(\gamma)\widetilde{Q}(\gamma)Z^{-1/2}(\gamma)\Big) \\ &\quad - 2\rho\sqrt{\bar{z}_{n+q,n+q}\lambda_{\max}(Z(\gamma))}\Bigg)\left\|Z^{-1/2}(\gamma)e\right\|^2 \\ &\quad + 2\|e'Z^{-1/2}(\gamma)\|\|Z^{-1/2}(\gamma)B\||w|,\end{aligned}$$

where $\bar{z}_{n+q,n+q}$ is the $(n+q, n+q)$ element of the matrix $Z^{-1}(\gamma)$. Under the assumption that γ is selected to satisfy the condition

$$\begin{aligned}C_1 = \lambda_{\min}&\Big(Z^{-1/2}(\gamma)\widetilde{Q}(\gamma)Z^{-1/2}(\gamma)\Big) \\ &- 2\rho\sqrt{\bar{z}_{n+q,n+q}\lambda_{\max}(Z(\gamma))} > 0,\end{aligned} \tag{44}$$

then, we have

$$\dot{V}(e) \le -C_1V(e) + 2\sqrt{V(e)}\left\|Z^{-1/2}(\gamma)B\right\||w|. \tag{45}$$

One can prove that

$$Z(\gamma) = \frac{1}{\gamma}D(\gamma)\widetilde{Z}D(\gamma), \tag{46}$$

where $\widetilde{Z} = Z(\gamma)|_{\gamma=1}$ and

$$D(\gamma) = \text{diag}\big[\gamma, \gamma^2, \ldots, \gamma^{n+q}\big]. \tag{47}$$

This gives after putting $W(e) = \sqrt{V(e)}$

$$\dot{W}(e) \le -\frac{C}{2}W(e) + \frac{C_2}{\gamma^{q/2}}|w|, \tag{48}$$

where $C_2 = \lambda_{\max}(\widetilde{Z}^{-1})$. If the integration order q increases then the observation error becomes smaller and smaller, which implies that for any SISO globally Lipschitz nonlinear system, written in Brunovski form, there exists always a robust observer that can filter out the estimates with a level $1/\gamma^{q/2}$. Further, if noise is absent the convergence is exponential, see (48). Inequality (48) characterizes also the input to state stability of the observation error with respect to the additive noise, see Sontag (1995) for more details. We have proved the following.

Corollary 2: *Consider system* (38). *Then under the fulfilment of condition* (44), *system* (40) *is a robust observer for system* (38) *that decouples the effect of noisy measurements from the observer gain. Furthermore, if* $w \equiv 0$, *the observation error is globally exponentially stable.*

Remark that for high values of γ, noise cannot be amplified if the order of integration q is selected in accordance to the fixed value γ, see (48). Hence, the order of integration q and γ may act simultaneously as key parameters to reduce the effect of noisy measurements. Notice also that γ plays a fundamental role in characterizing the transient behavior of the observer states. For this reason, the parameter γ should be selected according to the value of the Lipschitz constant. In the next section, discrete-time implementation of the observer discussed previously is given.

3. Discrete-time implementation of the high-gain observer

In order to implement the nonlinear observer developed in the last sections, two possible ways can be followed. The first possible way is to construct a continuous-time observer as shown in section 2, then discretize the continuous-time observer. The second way is to give a discretization of the continuous-time system and then build an observer for the approximate system. Preliminary discussion of discrete-time implementation of high-gain differentiation observers can be found in Dabroom and Khalil (1999) where no nonlinearities have been considered. In this section, we will discuss the discretization of high-gain observers in presence of Lipschitzian terms and show how to implement a discrete-time high-gain observer if the continuous-time observer is already designed. The major difficulty that presents itself in this case is how to choose properly the sampling period such that the states of the discretized observer converge asymptotically to the states of the Euler discretization of the continuous-time system.

System (3) is rewritten as

$$\left.\begin{aligned}\dot{x} &= A\,x + B\,\phi(x,u), \\ y &= C\,x.\end{aligned}\right\} \tag{49}$$

The Euler discretization of system (49) gives

$$\left.\begin{aligned}x_{k+1} &= A_\delta\,x_k + \delta\,B\,\phi(x_k, u_k), \\ y_k &= C\,x_k,\end{aligned}\right\} \tag{50}$$

where $x_k = x(k\delta)$, $k \in \mathbb{Z}_{\geq 0}$ is the discrete-time state vector, δ is the sampling period, and $A_\delta = \boldsymbol{I} + \delta A$. First, we shall look for sufficient conditions such that the states of the following observer

$$\hat{x}_{k+1} = A_\delta \hat{x}_k + \delta B \phi(x_k, u_k) + X^{-1} C'(y_k - C\hat{x}_k), \quad (51)$$

converge asymptotically to the states of system (50). The proposed observer can also be seen as the Euler discretezation of the observer

$$\dot{\hat{x}} = A\hat{x} + B\phi(\hat{x}, u) + P(\gamma)C'(y - C\hat{x}), \quad (52)$$

where A, B, C, and $P(\gamma)$ are defined as in section 2. The Euler discretization of observer (52) leads to the following discrete-time system

$$\hat{x}_{k+1} = A_\delta \hat{x}_k + \delta B\phi(\hat{x}_k, u_k) + \delta P(\gamma)C'(y_k - C\hat{x}_k). \quad (53)$$

The last difference system coincides with observer (51), if and only if $X^{-1} = \delta P(\gamma)$. Let $e_k = \hat{x}_k - x_k$ be the error between the states of systems (51) and (50). Then

$$e_{k+1} = (A_\delta - X^{-1}C'C)\, e_k + \delta B(\phi(\hat{x}_k, u_k) - \phi(x_k, u_k)). \quad (54)$$

Since the nonlinearity $\phi(\hat{x}_k, u_k)$ is supposed to be globally Lipschitz, then we can always find constant matrices M, and N such that

$$\mathscr{F}(s_k, u_k) = \delta B \frac{\partial \phi(x_k, u_k)}{\partial x_k} = \delta MF(s_k, u_k)N, \quad (55)$$

where $F'(s_k, u_k)F(s_k, u_k) < \boldsymbol{I}$ for all $x_k \in \mathscr{M}$ and $u_k \in \mathscr{U}$. The last difference equation (54) can be rewritten as

$$\begin{aligned} e_{k+1} &= \big(A_\delta - X^{-1}C'C\big)e_k + \int_0^1 \mathscr{F}(s_k, u_k)|_{s_k = \hat{x}_k - \lambda e_k}\, e_k \mathrm{d}\lambda \\ &= \int_0^1 \big(A_\delta - X^{-1}C'C + \mathscr{F}(s_k, u_k)|_{s_k = \hat{x}_k - \lambda e_k}\big)e_k \mathrm{d}\lambda. \quad (56) \end{aligned}$$

If we put $V_k = e_k' X e_k$ as a Lyapunov function candidate to (56), then we obtain

$$\begin{aligned} V_{k+1} - V_k &= \left[\int_0^1 e_k'\big(A_\delta' - C'CX^{-1} + \mathscr{F}'(s_k, u_k)|_{s_k = \hat{x}_k - \lambda e_k}\big)\mathrm{d}\lambda\right]X \\ &\quad \times \left[\int_0^1 \big(A_\delta + X^{-1}C'C + \mathscr{F}(s_k, u_k)|_{s_k = \hat{x}_k - \lambda e_k}\big)e_k\right] \\ &\quad - \int_0^1 e_k' X e_k \,\mathrm{d}\lambda. \quad (57) \end{aligned}$$

By the use of result of Lemma 2, we can write that

$$\begin{aligned} V_{k+1} - V_k &\leq \int_0^1 e_k'\big[\big(A_\delta' - C'CX^{-1} + \delta N'F'(s_k, u_k)|_{s_k = \hat{x}_k - \lambda e_k} M'\big)X \\ &\quad \times \big(A_\delta - X^{-1}C'C + \delta MF(s_k, u_k)|_{s_k = \hat{x}_k - \lambda e_k} N\big) - X\big]e_k \mathrm{d}\lambda. \quad (58) \end{aligned}$$

By the Schur complement lemma, a sufficient condition to make $V_{k+1} - V_k < 0$ is

$$\begin{bmatrix} -X & A_\delta' X - C'C + \int_0^1 \delta N'F'(s_k, u_k)|_{s_k = \hat{x}_k - \lambda e_k} M'X \mathrm{d}\lambda \\ \star & -X \end{bmatrix} < 0, \quad (59)$$

or

$$\begin{bmatrix} -X & A_\delta' X - C'C \\ \star & -X \end{bmatrix} + \int_0^1 \begin{bmatrix} \mathbf{0} \\ \delta XM \end{bmatrix} F(s_k, u_k)|_{s_k = \hat{x}_k - \lambda e_k} \begin{bmatrix} N' & \mathbf{0} \end{bmatrix} \mathrm{d}\lambda + \int_0^1 \begin{bmatrix} N' \\ \mathbf{0} \end{bmatrix} F'(s_k, u_k)|_{s_k = \hat{x}_k - \lambda e_k} \begin{bmatrix} \mathbf{0} & \delta M'X \end{bmatrix} \mathrm{d}\lambda < 0.$$

Using the fact that for given symmetric matrices Z_1 and Z_2 of appropriate dimensions, we have

$$Z_1' Z_2 + Z_2' Z_1 \leq \epsilon Z_1' Z_1 + \frac{1}{\epsilon} Z_2' Z_2, \quad (60)$$

for any $\epsilon > 0$, then

$$\begin{aligned} &\int_0^1 \begin{bmatrix} \mathbf{0} \\ \delta XM \end{bmatrix} F(s_k, u_k)|_{s_k = \hat{x}_k - \lambda e_k} \begin{bmatrix} N' & \mathbf{0} \end{bmatrix} \mathrm{d}\lambda \\ &\quad + \int_0^1 \begin{bmatrix} N' \\ \mathbf{0} \end{bmatrix} F'(s_k, u_k)|_{s_k = \hat{x}_k - \lambda e_k} \begin{bmatrix} \mathbf{0} & \delta M'X \end{bmatrix} \mathrm{d}\lambda \\ &\quad \leq \frac{1}{\epsilon} \begin{bmatrix} \mathbf{0} \\ \delta XM \end{bmatrix} \begin{bmatrix} \mathbf{0} & \delta M'X \end{bmatrix} \\ &\quad + \epsilon \begin{bmatrix} N' \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} N & \mathbf{0} \end{bmatrix}. \end{aligned}$$

This implies that if the following linear matrix inequality holds

$$\begin{bmatrix} -X + \epsilon N'N & A_\delta' X - C'C \\ XA_\delta - C'C & -X + \delta^2/\epsilon\, XMM'X \end{bmatrix} < 0, \quad (61)$$

then $V_{k+1} - V_k < 0$. Inequality (61) is equivalent by the Schur complement to the following LMI

$$\begin{bmatrix} -X + \epsilon N'N & A'_\delta X - C'C & 0 \\ XA_\delta - C'C & -X & \delta XM \\ 0 & \delta M'X & -\epsilon \boldsymbol{I} \end{bmatrix} < 0. \quad (62)$$

If for a given sampling period δ there exist $X = X' > 0$ of appropriate dimensions and a positive scalar ϵ such that the linear matrix inequality (62) holds then, the states of observer (51) converges asymptotically to the discrete states issued from the Euler discretization of the original system (49). Thanks to the sufficient linear matrix inequality condition (62) and based on results of Theorem 5, one can test the stability of a given Euler discretization of a continuous-time observer. We arrive at the following statement.

Theorem 2: *Consider system* (3) *and let $P(\gamma)$ be the solution of the algebraic Riccati equation* (6) *for γ satisfying condition of Theorem* 1. *Then if for a given sampling period δ there exists a constant $\tau > 0$ such that*

$$\begin{bmatrix} -\frac{1}{\delta}P^{-1}(\gamma) + \tau N'N & \frac{1}{\delta}A'_\delta P^{-1}(\gamma) - C'C & \mathbf{0} \\ \frac{1}{\delta}P^{-1}(\gamma)A_\delta - C'C & -\frac{1}{\delta}P^{-1}(\gamma) & P^{-1}(\gamma)M \\ \mathbf{0} & M'P^{-1}(\gamma) & -\tau\boldsymbol{I} \end{bmatrix} < 0. \quad (63)$$

Then the discrete-time system

$$\hat{x}_{k+1} = A_\delta\,\hat{x}_k + \delta\,B\phi(x_k, u_k) + \delta P(\gamma)(y_k - C\hat{x}_k), \quad (64)$$

is an asymptotic observer for system issued from Euler discretization of system (3).

Proof: The proof of Theorem 2 is already done by replacing X in (62) by the gain $(1/\delta)P^{-1}(\gamma)$.

The most interesting question that can be asked by observer designers concerns the maximum allowable sampling period that makes inequality (63) verified. This task is hopefully possible by replacing $1/\delta$ in (63) by a certain positive constant α and considering the following linear optimization problem

$$\min_{\tau} \alpha$$

s.t.

$$\begin{bmatrix} -\alpha P^{-1}(\gamma) + \tau N'N & (A' + \alpha\boldsymbol{I})P^{-1}(\gamma) - C'C & \mathbf{0} \\ (A + \alpha\boldsymbol{I})P^{-1}(\gamma) - C'C & -\alpha P^{-1}(\gamma) & P^{-1}(\gamma)M \\ \mathbf{0} & M'P^{-1}(\gamma) & -\tau\boldsymbol{I} \end{bmatrix} < 0. \quad (65)$$

Digital implementation of the robust observer (40) is identical. It is sufficient to replace in inequality (65) the matrix A by the augmented matrix $\tilde{A}$ and $P(\gamma)$ by $Z(\gamma)$.

4. Example

4.1. *Pendulum system*

After a particular choice of time-scale, equations of motions for the inverted pendulum can be written as follows

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \sin(x_1) + u\cos(x_1), \\ y &= x_1, \end{aligned}\right\} \quad (66)$$

where u is the normalized acceleration of the pivot, x_1 is the pendulum angle, and x_2 stands for the angular velocity. Following equations (6), we propose the high-gain observer

$$\left.\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + \alpha_1\gamma(y - \hat{x}_1), \\ \dot{\hat{x}}_2 &= \sin(\hat{x}_1) + u\cos(\hat{x}_1) + \alpha_2\gamma^2(y - \hat{x}_1). \end{aligned}\right\} \quad (67)$$

For this system we fix $\alpha_1 = 3$ and $\alpha_2 = 2$, which gives the Hurwitz polynomial $s^2 + 3s + 2$, having $s = -1$, and $s = -2$ as poles. Then we deduce that

$$Q(\gamma) = \begin{bmatrix} 5\gamma^2 & 0 \\ 0 & 4\gamma^4 \end{bmatrix},$$
$$P(\gamma) = \begin{bmatrix} 3\gamma & 2\gamma^2 \\ 2\gamma^2 & 6\gamma^3 \end{bmatrix},$$
$$P^{-1}(\gamma) = \begin{bmatrix} \frac{3}{7}\frac{1}{\gamma} & -\frac{1}{7}\frac{1}{\gamma^2} \\ \frac{1}{7}\frac{1}{\gamma^2} & \frac{3}{14}\frac{1}{\gamma^3} \end{bmatrix}. \quad (68)$$

Then

$$\bar{p}_{n,n}(\gamma) = \frac{3}{14}\frac{1}{\gamma^3},$$
$$\lambda_{\max}(P(\gamma)) = \frac{3}{2}\gamma + 3\gamma^3 + \frac{1}{2}\sqrt{36\gamma^6 - 20\gamma^4 + 9\gamma^2}. \quad (69)$$

This gives

$$\sqrt{\bar{p}_{n,n}(\gamma)\lambda_{\max}(P(\gamma))} = \sqrt{\frac{3}{28}\left(6 + \frac{3}{\gamma^2} + \sqrt{36 - \frac{20}{\gamma^2} + \frac{9}{\gamma^4}}\right)}. \quad (70)$$

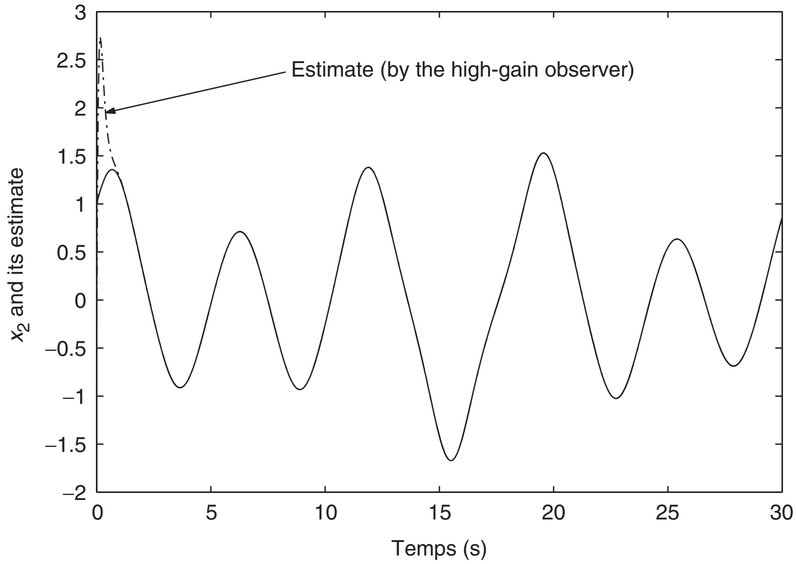

Figure 1. The second state x_2 and its estimate $\hat{x}_2$ given by the high-gain observer.

Remark that $\lim_{\gamma\to\infty}\sqrt{\bar{p}_{n,n}(\gamma)\lambda_{\max}(P(\gamma))}$ is independent of γ. Furthermore, because the nonlinearity $f(x_1,u)=\sin(x_1)+u\cos(x_1)$ is Lipschitz for any bounded control u, then for an excitation $u=0.5\sin(t)$, we can take $\rho=3/2$ as Lipschitz constant, and hence, we could finally fix the value of γ at 6 which gives

$$\left.\begin{aligned} &P^{-1/2}=\begin{bmatrix}0.2669 & -0.0134\\ -0.0134 & 0.0285\end{bmatrix},\\ &\lambda_{\min}(P^{-1/2}Q(\gamma)P^{-1/2})=3.5620,\\ &2\sqrt{\bar{p}_{n,n}\lambda_{\max}(P)}=2.2713,\end{aligned}\right\} \tag{71}$$

and verifies $\rho<1.5683$.

The estimate $\hat{x}_2$ is depicted in figure 1. In figure 2, we presented the noisy output, and in figure 3, we depicted the second estimated state given by the robust observer (40) under the action of the controller $u=0.5\sin(t)$. For this simulation the order of integration is set to 2, and the coefficients $\alpha_k=C_4^k$.

Now we discuss the availability of discrete-time observer gains for some prescribed sampling periods. The Jacobian of the nonlinearity can be rewritten as

$$\delta\,\frac{\partial f(x_k,u_k)}{\partial x_k}=\delta MF(x_k,u_k)N,$$

where $u_k=0.5\sin t_k$, $\rho=3/2$, and

$$M=\begin{bmatrix}0 & 0\\ 0 & \sqrt{\rho}\end{bmatrix},\ N=\begin{bmatrix}\sqrt{\rho} & 0\\ 0 & 0\end{bmatrix},$$
$$F(x_k,u_k)=\begin{bmatrix}0 & 0\\ 1/\rho\Big(\cos(x_k^{(1)})-u_k\sin(x_k^{(1)})\Big) & 0\end{bmatrix}.$$

Starting from the solutions (68), we can discretize the resulting continuous-time observer with a maximum sampling period $\delta=0.0915$. For this sampling step, the solution of the LMI (63) with the LMI package of MATLAB gives $\tau=0.0347$. This implies that for any sampling period $\delta\leq 0.0915$, the states of the discretized observer (53) converge asymptotically to the state of the Euler discretization of the continuous-time system.

The observer gain can also be recomputed using LMI (62). The solution of LMI (62) with respect to X, and ϵ gives for $\delta=0.01$

$$\epsilon=0.5671,\quad X=\begin{bmatrix}1.5533 & -0.1662\\ -0.1662 & 0.8352\end{bmatrix}.$$

Notice that this LMI remains solvable until a maximum sampling period $\delta=0.525$ where we get for this sampling period

$$\epsilon=0.3415,\quad X=\begin{bmatrix}1.1685 & -0.3414\\ -0.3414 & 0.3119\end{bmatrix}.$$

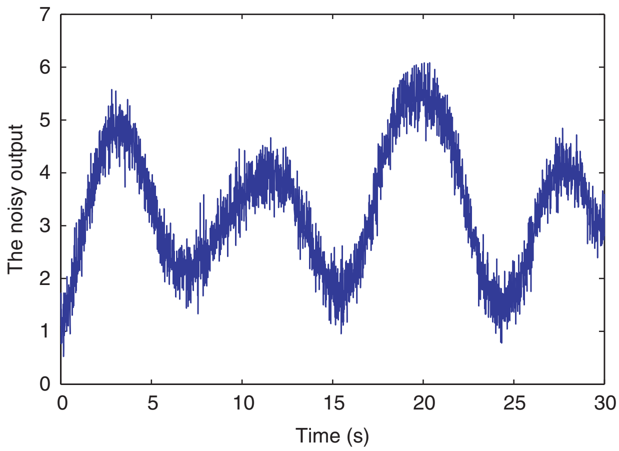

Figure 2. The noisy output.

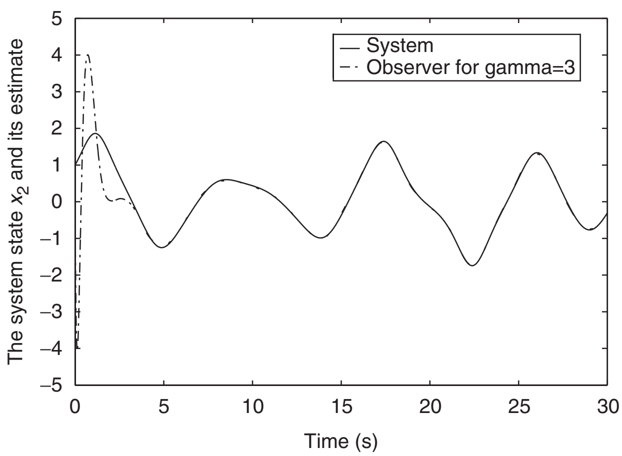

Figure 3. The second state x_2 and its estimate $\hat{x}_2$ given by the robust high-gain observer.

It is important to point out that the choice of the sampling period is also dependent upon the bandwidth of the states. In other words, the requirements of Shannon's theorem must also be checked.

5. Conclusions

In this paper robust high-gain nonlinear observers are discussed in continuous-time and discrete-time cases. The first observer is formulated as a classical Kalman observer, where the observer gain is calculated through a parameter-dependent Riccati equation. Optimality of such an observer is highlighted in terms of an integral inequality constraint. For systems that can be transformed into observable canonical form, we showed that the convergence of the high-gain observer is always guaranteed by increasing the value of a design parameter. The second proposed observer is a q-integral observer that permits noise reduction for high gain observer values. It is proved and shown that the robust observer furnishes nice filtered converging estimates in comparison with classical Luenberger observers. Practical discrete-time implementation of the developed observers is discussed in a linear matrix inequality framework.

Acknowledgements

The author would like to thank anonymous reviewers for their helpful comments. A part of this work had been done while the author was in the Department of Automated Production, École de Technologie Supérieure, 1100, rue Notre Dame Ouest, Montréal, Québec, Canada H3C 1K3.

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Salim Ibrir is currently a postdoctoral fellow in Concordia University, Montreal, Canada. He received his B.Eng. degree from Blida Institute of Aeronautics, Algeria, in 1991, and the Ph.D. degree from Paris-11 University, in 2000. From 1999 to 2000, he was a research associate (ATER) in the Department of Physics of Paris-11 University. In 2003 Dr. Ibrir was a lecturer in the Department of Automated Production of École de Technologie Supérieure, Montreal, Canada. He spent more than two years as postdoctoral fellow and visiting assistant professor in diverse north American universities. His current research interests are in the areas of nonlinear observers, nonlinear control of non-smooth systems, robust system theory and applications, time delay systems, hybrid systems, signal processing, ill-posed problems in estimation, singular hybrid systems and Aero-Servo-Elasticity. Dr. Ibrir is the author of more than 50 technical research papers and one text book.