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Observer design for discrete-time systems subject to time-delay nonlinearities

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In this paper, we address the problem of designing nonlinear observers for dynamical discrete-time systems with both constant and time-varying delay nonlinearities. The nonlinear system is assumed to verify the usual Lipschitz condition that permits us to transform the nonlinear system into a linear time-delay system with structured uncertainties. The existence of the observer-gain is ensured by the solution of a one linear matrix inequality. An illustrative example is included to demonstrate the advantage of the proposed observation technique.

Keywords: Nonlinear observers; Discrete-time systems; Time-varying delay systems; Linear matrix inequalities (LMIs)

1. Introduction

Time-delay systems have received widespread attention during the last decades due to their importance in control engineering practice (Hale and Lunel 1993, Dugard and Verriest 1997, Gu *et al*. 2003). Delays are inherent in many existing physical systems and lead, in general, to some undesirable performances and frequently cause the loss of stability. In particular, time-delay is often used to characterize the effects of inertia phenomena, transportation and transmissions. This class of system can also be found in other disciplines and engineering fields such as economics and biology.

Stability and stabilizability of linear discrete-time systems with bounded uncertainties and time-delays have been the subject of numerous papers (see for example Fridman and Shaked (2005) and the references therein). However, observer design for nonlinear discrete time-delay systems has received little attention. Recently, observer design for continuous-time systems subject to linear delayed states and nonlinear output disturbances was studied in Wang *et al*. (2002). In the discrete-time case, state estimation for multiple delays linear system was discussed in Boutayeb (2001). To the best of our knowledge, the observation issue of discrete-time systems with time-varying delay nonlinearities has not been fully investigated which motivates the present work.

The idea of transforming a nonlinear system into observable canonical forms has been widely used in nonlinear observer design techniques. The existence of such diffeomorphisms is generally attached to solutions of inherently nonlinear partial differential equations and other geometric conditions that cannot always be verified by existing physical systems, see for example Lee and Nam (1991) and Ciccarella *et al*. (1993b). Among the simplest state observer schemes, the Luenberger observer (Luenberger 1971) is the most well known. This standard approach to solving the state observer problem, in the continuous-time case, is to use a copy of the observed system and to add some correction terms attenuating the difference of the outputs, see Luenberger (1971), Thau (1973), Ciccarella *et al*. (1993a), Raghavan and Hedrick (1994), Rajmani, (1998), Arcak and Kokotović (2001), Abohy *et al*. (2002) and Kreisselmeier and Engel (2003).

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Similar results have been developed in the discrete-time case, see e.g., Reif and Unberhauen (1999), Reif *et al.* (1999), Lee and Nam (1991) and Ciccarella *et al.* (1993b). Some technical problems arise in such Luenberger design, due to the fact that a constant-gain observer is used to stabilize the observation error dynamics and call for new strategies to reduce the conservatism of numerical methods that permit us to fix the right value of the observer gain. We refer the reader to the references (Raghavan and Hedrick 1994, Rajamani and Cho 1998, Abohy *et al.* 2002) for further details on how to characterize the relation between the distance to unobservability and the Lipschitz constants of nonlinearities.

By the development of interior point methods (Boyd *et al.* 1994), nonlinear observer design becomes extremely attached to solutions of some convex optimization problems. These convex optimization problems, known as linear matrix inequalities are powerful tools that permit, in the general case, to solve the observation issue for nonlinear dynamics as they appear in practice. Unfortunately, these numerical methods have permitted us to solve the estimation issue for some particular nonlinear systems as systems with Lipschtizian nonlinear dynamics and other systems with positive slope growths. Arcak and Kokotović (2001), proposed a new observer design methodology for continuous-time nonlinear systems where the design was basically founded on the principle of the circle criterion. Unfortunately, extension of the circle criterion observer design to discrete-time nonlinear systems has not been discussed and necessitates more elaboration. In the discrete-time case, the LMI-based techniques for observer design are quite few (see for example, Azemi and Yaz 1997). Other techniques as used in Raghavan and Hedrick (1994) for continuous-time systems have been related to the solvability of algebraic Riccati equation that does not always have a solution for large Lipschitz constants. In our opinion, severe major approximation of nonlinearities undoubtedly leads to conservative conditions and hence, observer design for this class of system remains a challenging issue.

Due to the delay effects and the presence of nonlinear terms in the system dynamics, classical existing Luenberger observers cannot be applied to delay systems, in general, and necessitates a complete redesign of the observer-gains. Motivated by our earlier results on observation of Lipschitz nonlinear systems, in this paper, we continue our investigation on observer design for discrete-time systems with time-delay Lipschitzian nonlinearities. The ambitious goal motivating this work is to significantly reduce the conservatism of existing results with an extension to nonlinear time-delay systems. By the use of a new formulation of the Lipschitz property, we show that observer design for a nonlinear system with time-delay Lipschtizian nonlinearities is equivalent to an observation problem of a linear time-delay system with structured uncertainties. In our methodology, we shall consider the nonlinearity vectors as well-defined perturbation terms which help us establish a less conservative condition that guarantees both existence and efficient computation of the observer gain. We subsequently extend the obtained results to time-varying delay nonlinear systems where the design will be accomplished through the solution of a linear matrix inequality. It is worthwhile mentioning that the design of the observer-gain is a delay-independent strategy, but the amount of delay is supposed to be known to conceive the nonlinear observer.

The rest of the paper is as follows. In section 2, the description of the system under consideration along with some preliminary definitions are presented. In section 3, the theory of the nonlinear observer is given. Observer design for discrete-time nonlinear systems subject to time-varying delays will be the subject of section 4. In section 5, a numerical example is provided to demonstrate the applicability of the developed results. Throughout this paper, the notations $A>0$, $A<0$ and $A\leq 0$ mean that the matrix A is positive definite, negative definite and negative semi-definite, respectively. We note by A' the matrix transpose of A. In matrix notation, I and 0 are used to notify an identity matrix and a null matrix of appropriate dimensions, respectively. $\mathbb{R}$ is the set of real numbers, $\mathbb{N}$ and $\mathbb{Z}$ stand for the sets of natural numbers and integer numbers, respectively. "$\star$" is used to notify a matrix element that is induced by transposition.

For the clarity of the paper statements, we recall the Schur complement lemma. The proof of this result can be found in Boyd *et al.* (1994).

Lemma 1 (The Schur complement lemma)**:** *Given constant matrices M, N, Q of appropriate dimensions where M and Q are symmetric, then $Q>0$ and $M+N'Q^{-1}N<0$ if and only if*

$$\begin{bmatrix} M & N' \\ N & -Q \end{bmatrix} < 0,$$

or equivalently

$$\begin{bmatrix} -Q & N \\ N' & M \end{bmatrix} < 0,$$

The following fact is frequently used in the proof of the main statement of this paper. We prefer herein to recall it.

Fact 1: For given matrices A_1, and A_2 with appropriate dimensions, we have

$$A_1'A_2 + A_2'A_1 \leq \varepsilon A_1'A_1 + \varepsilon^{-1}A_2'A_2.$$

and

$$A_1'A_2 + A_2'A_1 \leq A_1'P^{-1}A_1 + A_2'PA_2$$

where ε is any positive constant and P is an arbitrary symmetric positive definite matrix of appropriate dimensions.

The result of the following interesting lemma (Gu 2000) will be used in setting the proof of the main result of this paper.

Lemma 2: *For any constant symmetric matrix $M \in \mathbb{R}^{n\times n}$, $M = M' > 0$, scalar $\gamma > 0$, vector function ω: $[0, \gamma] \mapsto \mathbb{R}^n$ such that the integration in the following is well defined, we have*

$$\gamma\int_0^{\gamma}\omega'(\beta)M\omega(\beta)\mathrm{d}\beta \geq \left(\int_0^{\gamma}\omega(\beta)\mathrm{d}\beta\right)'M\left(\int_0^{\gamma}\omega(\beta)\mathrm{d}\beta\right). \quad (1.1)$$

2. System description and preliminaries

Consider the time-delay nonlinear system

$$\begin{cases} x_{k+1} = A\,x_k + A_d\,x_{k-d} + f(x_k) + g(x_{k-d}) + \varphi(y_k,\, u_k), \\ y_k \quad = C\,x_k, \end{cases} \quad (2.1)$$

where d is the amount of delay, $x_k \in \mathscr{M} \subset \mathbb{R}^n$ is the state vector, and $x_{k-d} \in \mathscr{M} \subset \mathbb{R}^n$ stands for the delayed state vector. We assume that $\mathscr{M}$ is a subset of $\mathbb{R}^n$ where the observation of the system states takes place. We suppose that $x_k = \psi_k$ for $-d \leq k \leq 0$ where ψ_k is a real-valued vector. The control input u_k is an m dimensional control vector, and $y_k \in \mathbb{R}^p$ is the system output. We assume that the pair (A, C) is detectable. Before giving the main result of this paper, let us introduce the following definitions.

Definition 1 (Observability)**:** We say that system (2.1) is "observable" if for all different initial states x_0, $\tilde{x}_0$, there exist an interval $\{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}^+$ and an admissible control u_k defined on $\{0, 1, \ldots, N-1\}$ such that the associated outputs $y(x_0, u_k)$ and $\tilde{y}_k(\tilde{x}_0, u_k)$ are not identically equal on $\{0, 1, \ldots, N-1\}$.

Definition 2 (Universal inputs)**:** The input u_k is said "universal" on $\{0, 1, \ldots, N-1\}$ if it distinguishes all initial states $(x_0, \tilde{x}_0)$ on $\{0, 1, \ldots, N-1\}$. System (2.1) is said to be "uniformly observable" if every admissible control u_k defined on $\{0, 1, \ldots, N-1\}$, is a universal one. We shall call $\mathscr{U}$ the set of all admissible control inputs that makes system (2.1) uniformly observable.

In order to complete the description of system (2.1), we assume that f: $\mathscr{M} \to \mathbb{R}^n$ and g: $\mathscr{M} \to \mathbb{R}^n$ are Lipschitz nonlinearities satisfying the following assumptions.

Assumption 1: We assume that $f(x_k)$ is a smooth real-valued nonlinearity that satisfies $f(0) = 0$, and

$$\frac{\partial f(x_k)}{\partial x_k} = M_f\,F(x_k)\,N_f, \quad \forall x_k, \quad (2.2)$$

where $M_f \in \mathbb{R}^{n\times n}$, $N_f \in \mathbb{R}^{n\times n}$ are well-defined real matrices, and $F(x_k) \in \mathbb{R}^{n\times n}$ is a norm-bounded matrix satisfying $F'(s_k)F(s_k) \leq I$, $\forall s_k$; $k \in \mathbb{Z}_{\geq 0}$.

Assumption 2: The vector $g(\cdot)$ is a smooth vector satisfying $g(0) = 0$, and

$$\frac{\partial g(x_k)}{\partial x_k} = M_g\,G(x_k)\,N_g, \quad \forall x_k, \quad (2.3)$$

where $M_g \in \mathbb{R}^{n\times n}$, $N_g \in \mathbb{R}^{n\times n}$ are completely known matrices, and $G(x_k) \in \mathbb{R}^{n\times n}$ is a norm-bounded matrix satisfying $G'(s_k)\,G(s_k) \leq I$, $\forall s_k$; $k \in \mathbb{Z}_{\geq 0}$.

Remark 1: System (2.1) could describe many dynamic processes as robot manipulators subject to transmission delays. If the terms $A_d\,x_{k-d}$ and $g(x_{k-d})$ disappear from (2.1), the resulting system will reduce to that studied in the reference Azemi and Yaz (1997).

Remark 2: The formulation of the Lipschitz condition was generally defined as

$$\|f(x_k) - f(\hat{x}_k)\| \leq \gamma_f\|x_k - \hat{x}_k\|, \quad \forall\big(x_k,\, \hat{x}_k\big) \in \mathbb{R}^n \times \mathbb{R}^n,$$

where $\gamma_f = \|M_fF(x_k)N_f\|_{\infty}$. However, the formulation of the Lipschitz properties (2.2) and (2.3) do not involve any approximation of the Jacobian matrices by their maximum Euclidean norms. Therefore, these important formulations of the Lipschitz properties shall help to derive less conservative conditions, especially when the nonlinearities $f(x_k)$ and $g(x_{k-d})$ have high Lipschitz constants.

Remark 3: The formulations (2.2) and (2.3) are not necessary conditions for the fullfilment of the Lipschitz property in the general case. However, if the differentiability of $f(x_k)$ and $g(x_{k-d})$ are imposed, then the formulations (2.2) and (2.3) become equivalent to the notion of the Lipschitz property.

Remark 4: System (2.1) can be made a non-delay system with state augmentation. But this operation, will not be of crucial help if the observer is combined with memoryless state feedback. In addition, the regroupment of nonlinearities in a unique vector will augment the value of the Lipschitz constant of the resulting nonlinearity and does not reduce the complexity of the observer design.

The breakdown of the nonlinear observer is detailed in the following section.

3. Observer design for constant delay nonlinear systems

The main objective of this section is to extend the design of a Luenberger nonlinear observer, within the framework of convex optimization, to time-delay nonlinear systems. To this aim, we propose an observer of the following form

$$\begin{cases} \hat{x}_{k+1} = A\,\hat{x}_k + A_d\,\hat{x}_{k-d} + f(\hat{x}_k) + g(\hat{x}_{k-d}) \\ \qquad + \varphi(y_k,\, u_k) + L(\hat{y}_k - y_k), \\ \hat{y}_k = C\,\hat{x}_k, \end{cases} \tag{3.1}$$

such that $\lim_{k\to\infty} \hat{x}_k - x_k = 0$, where $L \in \mathbb{R}^{n\times p}$ is the observer gain to be determined. The high-gain injection term in (3.1) shall be conceived not only to overcome the effect of the nonlinearity $f(x_k)$, but also to defeat the effect of the time-delay term $g(x_{k-d})$. We summarize the result in the following statement.

Theorem 1: *Consider system* (2.1) *under the action of an input signal $u_k \in \mathcal{U}$. If there exist a symmetric and positive definite matrix $X \in \mathbb{R}^{n\times n}$, a matrix $Y \in \mathbb{R}^{n\times p}$, and two positive constants ε_1, and ε_2 such that the following matrix inequality holds*

$$\begin{bmatrix} -X + Q + \varepsilon_1 N_f' N_f & 0 & A'X + C'Y' & 0 & 0 \\ \star & -Q + \varepsilon_2 N_g' N_g & A_d' X & 0 & 0 \\ \star & \star & -X & XM_f & XM_g \\ \star & \star & \star & -\varepsilon_1 I & 0 \\ \star & \star & \star & \star & -\varepsilon_2 I \end{bmatrix} < 0. \tag{3.2}$$

Then the following system

$$\begin{cases} \hat{x}_{k+1} = A\,\hat{x}_k + A_d\,\hat{x}_{k-d} + f(\hat{x}_k) + g(\hat{x}_{k-d}) \\ \qquad + \varphi(y_k,\, u_k) + X^{-1}Y(\hat{y}_k - y_k), \\ \hat{y}_k = C\,\hat{x}_k, \end{cases} \tag{3.3}$$

is an asymptotic observer for system (2.1).

Proof: Let $e_k = \hat{x}_k - x_k$ be the observation error. Then using the fact that

$$f(\hat{x}_k) - f(x_k) = \int_0^1 \frac{\partial f(s_k)}{\partial s_k}\bigg|_{s_k = \hat{x}_k - \lambda e_k} e_k \,\mathrm{d}\lambda, \tag{3.4}$$

and

$$g(\hat{x}_{k-d}) - g(x_{k-d}) = \int_0^1 \frac{\partial g(s_k)}{\partial s_k}\bigg|_{s_k = \hat{x}_{k-d} - \lambda e_{k-d}} e_{k-d}\,\mathrm{d}\lambda. \tag{3.5}$$

Then the dynamics of the observation error is

$$\begin{aligned} e_{k+1} &= (A + X^{-1}YC)e_k + A_d\, e_{k-d} \\ &\quad + \int_0^1 M_f F(s_k)\big|_{s_k = \hat{x}_k - \lambda e_k} N_f e_k \,\mathrm{d}\lambda \\ &\quad + \int_0^1 M_g G(s_k)\big|_{s_k = \hat{x}_{k-d} - \lambda e_{k-d}} N_g e_{k-d}\,\mathrm{d}\lambda. \end{aligned} \tag{3.6}$$

Let

$$\mathscr{K}(\hat{x}_k, e_k, \lambda) = M_f F(s_k) N_f\big|_{s_k = \hat{x}_k - \lambda e_k}, \tag{3.7}$$

$$\mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda) = M_g G(s_k) N_g\big|_{s_k = \hat{x}_{k-d} - \lambda e_{k-d}}, \tag{3.8}$$

and let us associate the Lyapunov Krasovskii functional

$$V_k = e_k' X e_k + \sum_{i=k-d}^{k-1} e_i' Q e_i, \tag{3.9}$$

to the dynamics of the observation error (3.6). Then $\Delta V_k = V_{k+1} - V_k$ is

$$\begin{aligned} \Delta V_k &= \Big[\int_0^1 \Big(A + X^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big)e_k \\ &\quad + \Big(A_d + \mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda)\Big)e_{k-d}\,\mathrm{d}\lambda\Big]' X \\ &\quad \times \Big[\int_0^1 \Big(A + X^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big)e_k \\ &\quad + \Big(A_d + \mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda)\Big)e_{k-d}\,\mathrm{d}\lambda\Big] \\ &\quad - e_k' X e_k + e_k' Q e_k - e_{k-d}' Q e_{k-d}. \end{aligned} \tag{3.10}$$

Using the result of lemma 2, then we have

$$\begin{aligned} \Delta V_k &\le \int_0^1 \Big[\Big(A + X^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big)e_k \\ &\quad + \Big(A_d + \mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda)\Big)e_{k-d}\Big]' X \\ &\quad \times \Big[\Big(A + X^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big)e_k \\ &\quad + \Big(A_d + \mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda)\Big)e_{k-d}\Big]\mathrm{d}\lambda \\ &\quad - \int_0^1 \Big[e_k' X e_k + e_k' Q e_k - e_{k-d}' Q e_{k-d}\Big]\mathrm{d}\lambda. \end{aligned} \tag{3.11}$$

This implies that

$$\begin{aligned} \Delta V_k &\le \int_0^1 e_k' \Big[\Big(A + X^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big)' \\ &\quad \times X\Big(A + X^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big) - X + Q\Big]e_k\,\mathrm{d}\lambda \\ &\quad + \int_0^1 e_k' \Big[\Big(A + X^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big)' \end{aligned}$$

$$\times X\Big(A_d + \mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda)\Big)\Big]e_{k-d}\,\mathrm{d}\lambda$$
$$+\int_0^1 e'_{k-d}\Big(A_d + \mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda)\Big)'$$
$$\times X\Big(A + X^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big)e_k\,\mathrm{d}\lambda$$
$$+\int_0^1 e'_{k-d}\Big\{\Big(A_d + \mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda)\Big)'$$
$$\times X\Big(A_d + \mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda)\Big) - Q\Big\}e_{k-d}\,\mathrm{d}\lambda$$
$$= \int_0^1 \begin{bmatrix} e_k \\ e_{k-d} \end{bmatrix}' \begin{bmatrix} \mathscr{B}_{1,1} & \mathscr{B}_{1,2} \\ \mathscr{B}_{2,1} & \mathscr{B}_{2,2} \end{bmatrix} \begin{bmatrix} e_k \\ e_{k-d} \end{bmatrix}$$
$$(\hat{x}_k, \hat{x}_{k-d}, e_k, e_{k-d}\lambda)\,\mathrm{d}\lambda, \tag{3.12}$$

where

$$\mathscr{B}_{1,1} = \Big(A + X^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big)'X$$
$$\times\Big(A + X^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big) - X + Q,$$
$$\mathscr{B}_{1,2} = \Big(A + X^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big)'X$$
$$\times\Big(A_d + \mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda)\Big),$$
$$\mathscr{B}_{2,1} = \mathscr{B}'_{1,2},$$
$$\mathscr{B}_{2,2} = \Big(A_d + \mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda)\Big)'X$$
$$\times\Big(A_d + \mathscr{L}(\hat{x}_{k-d}, e_{k-d}, \lambda)\Big) - Q.$$

The observation error is stable if

$$\int_0^1 \begin{bmatrix} \mathscr{B}_{1,1} & \mathscr{B}_{1,2} \\ \mathscr{B}_{2,1} & \mathscr{B}_{2,2} \end{bmatrix}(\hat{x}_k, \hat{x}_{k-d}, e_k, e_{k-d}, \lambda)\,\mathrm{d}\lambda < 0. \tag{3.13}$$

By the Schur complement lemma, inequality (3.13) is equivalent to

$$\int_0^1 \begin{bmatrix} Q - X & 0 & A' + C'Y'X^{-1} + \mathscr{K}' \\ \star & -Q & A'_d + \mathscr{L}' \\ \star & \star & -X^{-1} \end{bmatrix}$$
$$(\hat{x}_k, \hat{x}_{k-d}, e_k, e_{k-d}, \lambda)\,\mathrm{d}\lambda < 0 \tag{3.14}$$

or

$$\int_0^1 \begin{bmatrix} Q - X & 0 & A' + C'Y'X^{-1} \\ \star & -Q & A'_d \\ \star & \star & -X^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & N'_f F'(s_k)|_{s_k=\hat{x}_k-\lambda e_k} M'_f \\ \star & 0 & N'_g G'(s_k)|_{s_k=\hat{x}_{k-d}-\lambda e_{k-d}} M'_g \\ \star & \star & 0 \end{bmatrix}\mathrm{d}\lambda < 0.$$

The last matrix inequality can be rewritten as follows

$$\int_0^1 \left\{ \begin{bmatrix} Q - X & 0 & A' + C'Y'X^{-1} \\ \star & -Q & A'_d \\ \star & \star & -X^{-1} \end{bmatrix} + \begin{bmatrix} N'_f \\ 0 \\ 0 \end{bmatrix} F'(\hat{x}_k - \lambda e_k)\begin{bmatrix} 0 & 0 & M'_f \end{bmatrix} + \begin{bmatrix} 0 \\ N'_g \\ 0 \end{bmatrix} G'(\hat{x}_{k-d} - \lambda e_{k-d})\begin{bmatrix} 0 & 0 & M'_g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_f \end{bmatrix} F(\hat{x}_k - \lambda e_k)\begin{bmatrix} 0 & 0 & N_f \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_g \end{bmatrix} G(\hat{x}_{k-d} - \lambda e_{k-d})\begin{bmatrix} 0 & N_g & 0 \end{bmatrix} \right\}\mathrm{d}\lambda < 0. \tag{3.15}$$

Using fact 1, with $G'(\hat{x}_{k-d} - \lambda e_{k-d})G(\hat{x}_{k-d} - \lambda e_{k-d}) \le I$ and $F'(\hat{x}_k - \lambda e_k)F(\hat{x}_k - \lambda e_k) \le I$, we can write that a sufficient condition to make $\Delta V_k \le 0$ is

$$\int_0^1 \left\{ \begin{bmatrix} Q - X & 0 & A' + C'Y'X^{-1} \\ \star & -Q & A'_d \\ \star & \star & -X^{-1} \end{bmatrix} + \varepsilon_1 \begin{bmatrix} N'_f \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} N'_f \\ 0 \\ 0 \end{bmatrix}' + \frac{1}{\varepsilon_1}\begin{bmatrix} 0 \\ 0 \\ M_f \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ M_f \end{bmatrix}' + \varepsilon_2 \begin{bmatrix} 0 \\ N'_g \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ N'_g \\ 0 \end{bmatrix}' + \frac{1}{\varepsilon_2}\begin{bmatrix} 0 \\ 0 \\ M_g \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ M_g \end{bmatrix}' \right\}\mathrm{d}\lambda < 0, \tag{3.16}$$

or

$$\begin{bmatrix} Q - X + \varepsilon_1 N'_f N_f & 0 & A' + C'Y'X^{-1} \\ \star & -Q + \varepsilon_2 N'_g N_g & A'_d \\ \star & \star & -X^{-1} + \varepsilon_1^{-1} M_f M'_f + \varepsilon_2^{-1} M_g M'_g \end{bmatrix} < 0. \tag{3.17}$$

Inequality (3.17) can be rewritten as follows

$$\begin{bmatrix} I & 0 & 0 \\ \star & I & 0 \\ \star & \star & X^{-1} \end{bmatrix} \begin{bmatrix} Q - X + \varepsilon_1 N_f' N_f & 0 & A'X + C'Y' \\ \star & -Q + \varepsilon_2 N_g' N_g & A_d' X \\ \star & \star & -X + X(\varepsilon_1^{-1} M_f M_f' + \varepsilon_2^{-1} M_g M_g')X \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ \star & I & 0 \\ \star & \star & X^{-1} \end{bmatrix} < 0 \quad (3.18)$$

which implies that if

$$\begin{bmatrix} Q - X + \varepsilon_1 N_f' N_f & 0 & A'X + C'Y' \\ \star & -Q + \varepsilon_2 N_g' N_g & A_d' X \\ \star & \star & -X + X(\varepsilon_1^{-1} M_f M_f' + \varepsilon_2^{-1} M_g M_g')X \end{bmatrix} < 0 \quad (3.19)$$

is satisfied, then the observation error is asymptotically stable. The last matrix inequality is equivalent by the Schur complement to the linear matrix inequality (3.2). This ends the proof.

Condition (3.2) has been fixed through straightforward Lyapunov design without any major approximation of nonlinearities. This certainly reduces the conservatism of the LMI condition (3.2) and makes the computation of the observer gain an easy task by the use of available commercial LMI softwares.

As importantly, we have proved with the help of the new representations (2.2) and (2.3), that observer design for the discrete time-delay system (2.1) is equivalent to observer design for the following system

$$\left.\begin{aligned} x_{k+1} &= (A + \Delta A(x_k))\, x_k + (A_d + \Delta A_d(x_{k-d}))\, x_{k-d} \\ &\quad + \varphi(y_k, u_k), \\ y_k &= C x_k, \end{aligned}\right\} \quad (3.20)$$

where $\Delta A(x_k)$ and $\Delta A_d(x_{k-d})$ are defined as follows

$$\left.\begin{aligned} \Delta A(x_k) &= M_f\, E_f(x_k)\, N_f, \\ \Delta A_d(x_{k-d}) &= M_g\, E_g(x_{k-d})\, N_g, \end{aligned}\right\} \quad (3.21)$$

and $E_f(x_k) = \int_0^1 F(\lambda x_k)\,\mathrm{d}\lambda$, $E_g(x_{k-d}) = \int_0^1 G(\lambda\, x_{k-d})\,\mathrm{d}\lambda$. We would obtain the same condition of Theorem 1 if we construct an observer for (3.20) as

$$\left.\begin{aligned} \hat{x}_{k+1} &= \big(A + \Delta A(x_k)\big)\, \hat{x}_k + \big(A_d + \Delta A_d(x_{k-d})\big)\, \hat{x}_{k-d} \\ &\quad + \varphi(y_k, u_k) + P^{-1} Y(\hat{y}_k - y_k), \\ \hat{y}_k &= C \hat{x}_k, \end{aligned}\right\} \quad (3.22)$$

where $\Delta A(x_k)$ and $\Delta A_d(x_{k-d})$ are measurable perturbations terms. In conclusion, this important remark will serve as a starting point for establishing any stability breakdown generally encountered when the observer is used in closed-loop configuration.

Remark 5: If the nonlinearities $f(x_k)$ and $g(x_{k-d})$ are not globally Lipschitz, we can always consider a subset $\mathscr{M} \subset \mathbb{R}^n$ where the system nonlinearities can be made locally Lipschitz and hence, the developed results remain valid.

4. Extension to interval time-delay nonlinear systems

4.1 *Non-delayed output case*

In case of time-varying delay, the state augmentation technique cannot be applied to transform the time-varying delay system into a non-delay system. Therefore, the construction of a time-delay observer becomes an unavoidable task. In this section, we extend the obtained results to time-varying delay systems described by the following equations

$$\left\{\begin{aligned} x_{k+1} &= A\, x_k + A_d\, x_{k-d(k)} + f(x_k) + g(x_{k-d(k)}) + \varphi(y_k, u_k), \\ y_k &= C x_k, \end{aligned}\right. \quad (4.1)$$

where the nonlinearities $f(x_k)$ and $g(x_{k-d(k)})$ satisfy the previous assumptions (2.2) and (2.3), respectively. The nominal matrices $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{p\times n}$ are constants matrices, whereas $d(k)$ is a discrete-time delay that verifies $0 \le d(k) \le \bar{d}$, where $\bar{d}$ is a given positive integer. The main result of this section is given in the following statement.

Theorem 2: *Consider system* (4.1). *If there exist two matrices* $X = X' > 0$, $Q = Q' > 0$ *of dimensions* $n \times n$, *a matrix* Y *of dimensions* $n \times p$ *and two constants* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *such that the following* *linear matrix inequality holds*

$$\begin{bmatrix} -X+\varepsilon_1 N_f' N_f & 0 & A'(X+\bar{d}Q)+C'Y' & 0 & 0 \\ \star & -Q+\varepsilon_2 N_g' N_g & A_d'(X+\bar{d}Q) & 0 & 0 \\ \star & \star & -X-\bar{d}Q & (X+\bar{d}Q)M_f & (X+\bar{d}Q)M_g \\ \star & \star & \star & -\varepsilon_1 I & 0 \\ \star & \star & \star & \star & -\varepsilon_2 I \end{bmatrix} < 0. \tag{4.2}$$

Then the following discrete-time system

$$\left.\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + A_d\hat{x}_{k-d(k)} + f(\hat{x}_k) + g(\hat{x}_{k-d(k)}) \\ &\quad + \varphi(y_k, u_k) + (X+\bar{d}Q)^{-1}Y(\hat{y}_k - y_k), \\ \hat{y}_k &= C\hat{x}_k, \end{aligned}\right\} \tag{4.3}$$

is an asymptotic observer for (4.1) *for any input* $u_k \in \mathscr{U}$.

Proof: Define $e_k = \hat{x}_k - x_k$. Since

$$\begin{aligned} f(\hat{x}_k) - f(x_k) &= \int_0^1 \mathscr{K}(\hat{x}_k, e_k, \lambda)\, e_k \, d\lambda, \\ g(\hat{x}_{k-d(k)}) - g(x_{k-d(k)}) &= \int_0^1 \mathscr{G}(\hat{x}_{k-d(k)}, e_{k-d(k)}, \lambda)\, e_{k-d(k)}\, d\lambda, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \mathscr{K}(\hat{x}_k, e_k, \lambda) &= \left[\frac{\partial f(s_k)}{\partial s_k}\right]_{s_k=\hat{x}_k-\lambda e_k}, \\ \mathscr{G}(\hat{x}_{k-d(k)}, e_{k-d(k)}, \lambda) &= \left[\frac{\partial g(s_k)}{\partial s_k}\right]_{s_k=\hat{x}_{k-d(k)}-\lambda e_{k-d(k)}}. \end{aligned} \tag{4.5}$$

Then, we obtain

$$\begin{aligned} e_{k+1} &= (A + (X+\bar{d}Q)^{-1}YC)e_k + A_d e_{k-d(k)} \\ &\quad + \int_0^1 \mathscr{K}(\hat{x}_k, e_k, \lambda)\, e_k\, d\lambda \\ &\quad + \int_0^1 \mathscr{G}(\hat{x}_{k-d(k)}, e_{k-d(k)}, \lambda)\, e_{k-d(k)}\, d\lambda. \end{aligned} \tag{4.6}$$

Associating the Lyapunov-Krasovskii functional

$$V_k = e_k'(X+\bar{d}Q)e_k + \sum_{i=1}^{\bar{d}} \sum_{j=k-i}^{k-1} e_j' Q e_j, \tag{4.7}$$

to (4.6). This gives

$$\begin{aligned} V_{k+1} - V_k &= e_{k+1}'(X+\bar{d}Q)e_{k+1} \\ &\quad - e_k'(X+\bar{d}Q)e_k + \bar{d}e_k'Qe_k - \sum_{i=1}^{\bar{d}} e_{k-i}'Qe_{k-i}. \end{aligned} \tag{4.8}$$

Using the fact that

$$\sum_{i=1}^{\bar{d}} e_{k-i}'Qe_{k-i} \geq e_{k-d(k)}'Qe_{k-d(k)}, \ \forall k \in \mathbb{Z}, \tag{4.9}$$

we conclude that

$$\begin{aligned} V_{k+1} - V_k &\leq e_{k+1}'(X+\bar{d}Q)e_{k+1} - e_k'Xe_k \\ &\quad - e_{k-d(k)}'Qe_{k-d(k)}. \end{aligned} \tag{4.10}$$

Using (4.6) and (4.10), we can write that

$$\begin{aligned} \Delta V_k \leq & \Bigg[\int_0^1 \Big(A + (X+\bar{d}Q)^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big)e_k \\ & + \Big(A_d + \mathscr{G}(\hat{x}_{k-d(k)}, e_{k-d(k)}, \lambda)\Big)e_{k-d(k)}\, d\lambda\Bigg]'(X+\bar{d}Q) \\ & \times \Bigg[\int_0^1 \Big(A + (X+\bar{d}Q)^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\Big)e_k \\ & + \Big(A_d + \mathscr{G}(\hat{x}_{k-d(k)}, e_{k-d(k)}, \lambda)\Big)e_{k-d(k)}\, d\lambda\Bigg] \\ & - \int_0^1 \Big[e_k'Xe_k + e_{k-d(k)}'Qe_{k-d(k)}\Big] d\lambda. \end{aligned} \tag{4.11}$$

Using result of lemma 2, we have for any

$$\xi_k = \begin{bmatrix} e_k \\ e_{k-d(k)} \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$

we have

$$V_{k+1} - V_k \leq \int_0^1 \xi_k' \begin{bmatrix} \mathscr{S}_{1,1} & \mathscr{S}_{1,2} \\ \mathscr{S}_{2,1} & \mathscr{S}_{2,2} \end{bmatrix} \xi_k \, d\lambda, \tag{4.12}$$

where

$$\mathscr{S}_{1,1} = \left(A + (X + \bar{d}Q)^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\right)'(X + \bar{d}Q) \times \left(A + (X + \bar{d}Q)^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\right) - X,$$

$$\mathscr{S}_{1,2} = \left(A + (X + \bar{d}Q)^{-1}YC + \mathscr{K}(\hat{x}_k, e_k, \lambda)\right)' \times (X + \bar{d}Q)\left(A_d + \mathscr{G}(\hat{x}_{k-d(k)}, e_{k-d(k)}, \lambda)\right),$$

$$\mathscr{S}_{2,1} = \mathscr{S}'_{1,2},$$

$$\mathscr{S}_{2,2} = \left(A_d + \mathscr{G}(\hat{x}_{k-d(k)}, e_{k-d(k)}, \lambda)\right)'(X + \bar{d}Q) \times \left(A_d + \mathscr{G}(\hat{x}_{k-d(k)}, e_{k-d(k)}, \lambda)\right) - Q.$$

By analogy with (3.13), we find after straightforward development that a sufficient condition to ensure $V_{k+1} - V_k \leq 0$ is (4.2). The development is omitted here since it is quite similar to that developed in section 3. This ends the proof.

The main contribution of this paper is basically founded on how one could transform the problem of observer design for discrete-time delay nonlinear system into a robust stability problem of a linear system with structured uncertainties. We have shown that neither the state augmentation procedure nor the bounding technique for cross terms is involved in derivation of the observation error stability condition. This certainly reduces the conservatism of the stability criteria, however, the time-varying delay case imposes additional conservatism which renders the proposed observer design quite limited for small changes in time-delay.

4.2 *Delayed output case*

If other Lipschitzian nonlinearities are present in the output equation as $y_k = Cx_k + h_1(x_k) + h_2(x_{k-d})$, we can always translate the nonlinearities $h_1(x_k)$ and $h_2(x_{k-d})$ to the state dynamics by system augmentation with the state z_k such that $z_{k+1} = Dz_k + Cx_k + h_1(x_k) + h_2(x_{k-d})$ i.e.,

$$\left.\begin{aligned} x_{k+1} &= A\,x_k + A_d\,x_{k-d(k)} + f(x_k) + g(x_{k-d(k)}) + \varphi(y_k, u_k),\\ z_{k+1} &= D\,z_k + Cx_k + h_1(x_k) + h_2(x_{k-d(k)}),\\ \tilde{y}_k &= \begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} x_k \\ z_k \end{bmatrix}, \end{aligned}\right\} \tag{4.13}$$

where z_k is considered as the new output. Consequently, the resulting system takes again the form of (4.1).

The matrix $D \in \mathbb{R}^{p\times p}$ is chosen to make the system $z_{k+1} = Dz_k$ asymptotically stable. In addition, D must also be fixed in order to guarantee the observability of the pair

$$\left(\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}, \begin{bmatrix} 0 & I \end{bmatrix}\right). \tag{4.14}$$

For this specific case, we see that the simplicity of the observer design avoids another synthesis since the computation of the new observer gain will be based only on both the gradients of nonlinearities and the nominal matrices that result from the state augmentation.

5. Illustrative examples

In this section, we present two illustrative examples. In the first example, we confirm the results obtained through numerical simulation of the discrete-time delay observer (3.3). In the second example, we highlight that the proposed linear matrix inequality condition that guarantees the existence of the observer is not conservative. To test the flexibility and the numerical tractability of (3.2) when the Lipschitz constants of the system nonlinearities are notably high, we increase gradually the values of the Lipschitz constants and show that the LMI (3.2) is solvable.

5.1 *First example*

5.1.1 Constant delay case. Consider the following nonlinear time-delay system

$$\left.\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.9 & -0.297 & 0.45 \\ -1.494 & 0.18 & 0.9 \\ 0.9 & -0.9 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0.1 & 0.02 & 0.1 \\ -0.2 & 0 & 0.1 \\ 0.1 & -0.2 & 0 \end{bmatrix} x_{k-d} \\ &+ \begin{bmatrix} -\frac{1}{12}\left(\cos(x_k^{(1)}) - 1\right) \\ \frac{1}{16}x_k^{(2)} - \frac{1}{16}\ln\left(1 + x_k^{(2)^2}\right) \\ \frac{1}{20}\sin\left(x_k^{(3)}\right) \end{bmatrix} + \begin{bmatrix} \frac{1}{8}\arctan\left(x_{k-d}^{(1)}\right) \\ 0 \\ \frac{1}{8}\arctan\left(x_{k-d}^{(3)}\right) \end{bmatrix} \\ &- \frac{1}{40}\begin{bmatrix} \left(x_k^{(2)} + x_k^{(3)}\right)\left(x_k^{(1)} + x_k^{(3)}\right) \\ \left(x_k^{(2)} + x_k^{(3)}\right)u_k \\ \left(x_k^{(1)} + x_k^{(3)}\right)u_k \end{bmatrix}, \\ y_k &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x_k. \end{aligned}\right\} \tag{5.1}$$

Define

$$f(x_k) = \begin{bmatrix} -\frac{1}{12}\left(\cos(x_k^{(1)}) - 1\right) \\ \frac{1}{16} x_k^{(2)} - \frac{1}{8}\ln\left(1 + x_k^{(2)^2}\right) \\ \frac{1}{20}\sin\left(x_k^{(3)}\right) \end{bmatrix},$$

$$g(x_{k-d}) = \begin{bmatrix} \frac{1}{8}\arctan\left(x_{k-d}^{(1)}\right) \\ 0 \\ \frac{1}{8}\arctan\left(x_{k-d}^{(3)}\right) \end{bmatrix},$$

$$\varphi(y,u) = -\frac{1}{40}\begin{bmatrix} \left(x_k^{(2)} + x_k^{(3)}\right)\left(x_k^{(1)} + x_k^{(3)}\right) \\ \left(x_k^{(2)} + x_k^{(3)}\right)u_k \\ \left(x_k^{(1)} + x_k^{(3)}\right)u_k \end{bmatrix}.$$

Then

$$M_f = \text{diag}[1/6,\ 1/2,\ 1/10],\ N_f = \text{diag}[1/2,\ 1/2,\ 1/2],$$
$$M_g = \text{diag}[1/4,\ 0,\ 1/4],\ N_g = M_g,$$

$$F(x_k) = \begin{bmatrix} \sin\left(x_k^{(1)}\right) & 0 & 0 \\ 0 & \frac{1}{4} - \frac{1}{2}\frac{x_k^{(2)}}{1 + x_k^{(2)^2}} & 0 \\ 0 & 0 & \cos\left(x_k^{(3)}\right) \end{bmatrix},$$

$$G(x_k) = \text{diag}\left[1/\left(1 + x_k^{(1)^2}\right),\ 0,\ 1/\left(1 + x_k^{(3)^2}\right)\right].$$

We have used the LMI package of Matlab to solve the linear matrix inequality (3.2). The solution of this LMI gives

$$X = \begin{bmatrix} 1.8368 & 0.1429 & -0.3498 \\ 0.1429 & 0.4691 & -0.0535 \\ -0.3498 & -0.0535 & 1.6649 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.7371 & -0.0133 & 0.0310 \\ -0.0133 & 0.1834 & 0.0448 \\ 0.0310 & 0.0448 & 0.6906 \end{bmatrix},$$

$$Y = \begin{bmatrix} -1.1434 & 0.2040 \\ 0.1418 & -0.2111 \\ -1.2302 & 1.4076 \end{bmatrix},$$

$$\varepsilon_1 = 0.6772, \quad \varepsilon_2 = 0.8837.$$

Starting form the initial conditions $x_0 = [-10\ 10\ \ -10]'$ with $u_k = \sin(10^3\, t_k)$, the history of the observer states along with the system states are represented in figure 1. For this simulation, the delay is set to $d=2$, and $\psi_k = 0$ for $-2 \leq k \leq 0$.

5.1.2 Time-varying delay case. In this case, we reconsider the same example (5.1) where the time delay d is replaced by time-varying delay that belongs to $\{1, 2, 3\}$. By solving the LMI (4.2) with respect to X, Y, Q, ε_1 and ε_2, we find for $\bar{d} = 3$

$$\left.\begin{aligned} X &= \begin{bmatrix} 0.9510 & 0.1247 & -0.2564 \\ 0.1247 & 0.1267 & -0.0518 \\ -0.2564 & -0.0518 & 0.3706 \end{bmatrix}, \\ Q &= \begin{bmatrix} 0.3117 & -0.0065 & 0.0070 \\ -0.0065 & 0.0556 & 0.0152 \\ 0.0070 & 0.0152 & 0.1988 \end{bmatrix}, \\ Y &= \begin{bmatrix} -1.2979 & 0.3353 \\ -0.0394 & -0.0906 \\ -0.6823 & 0.7994 \end{bmatrix}, \\ \varepsilon_1 &= 0.3958, \quad \varepsilon_2 = 0.4351. \end{aligned}\right\} \tag{5.2}$$

In order to show the performance of the time-varying observer, we have simulated the observer for different time-varying delays. In the first case, the delay $d(k)$ is defined as

$$d(k) = \begin{cases} 3 & \text{if } 0 \leq t_k \leq 0.08 \\ 2 & \text{if } 0.08 < t_k \leq 0.3 \\ 1 & \text{if } 0.3 < t_k \leq 0.4 \\ 2 & \text{if } 0.4 < t_k \leq 0.6 \\ 3 & \text{otherwise,} \end{cases} \tag{5.3}$$

and $x_k = 0$ for $k < 0$. The observer performance for this case is shown in figure 2. By changing $d(k)$ to the following

$$d(k) = \begin{cases} 1 & \text{if } 0 \leq t_k \leq 0.05 \\ 2 & \text{if } 0.05 < t_k \leq 0.2 \\ 1 & \text{if } 0.2 < t_k \leq 0.5 \\ 2 & \text{if } 0.5 < t_k \leq 0.6 \\ 3 & \text{otherwise.} \end{cases} \tag{5.4}$$

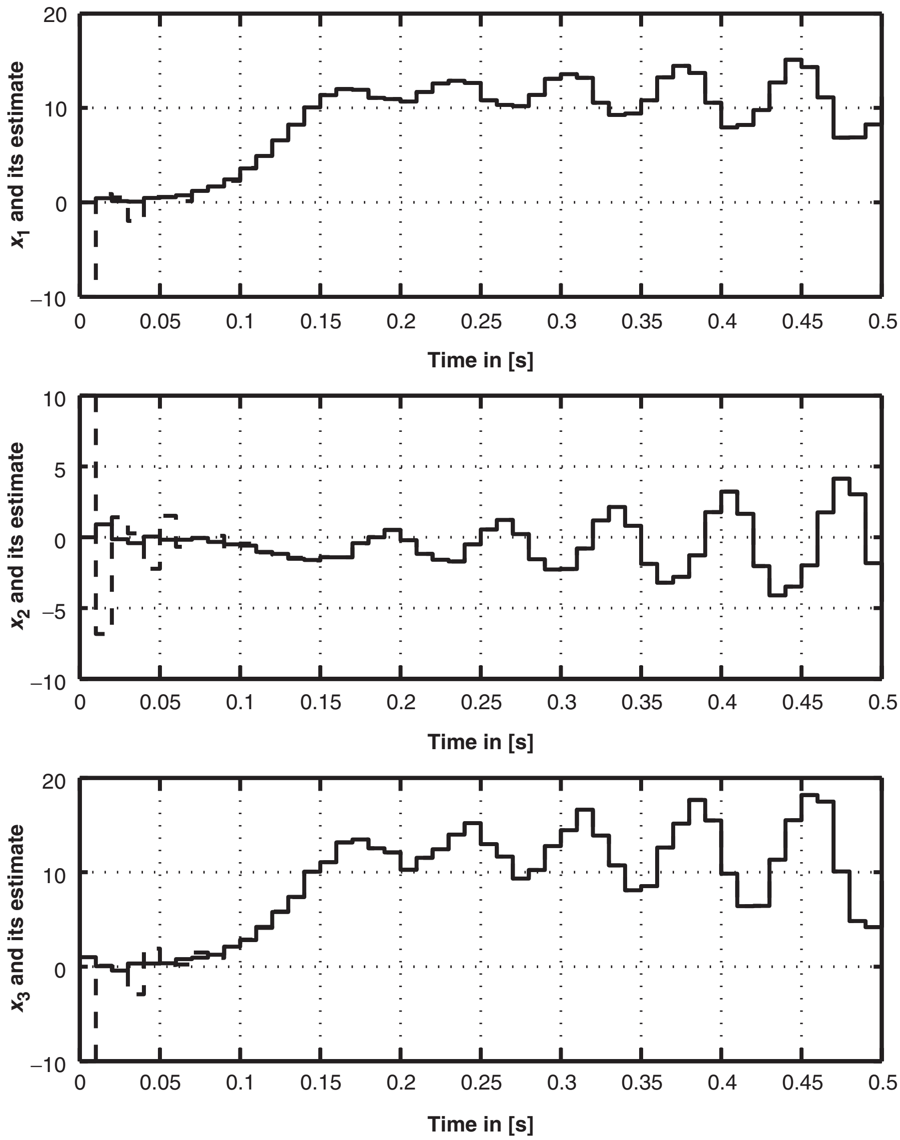

Figure 1. The performance of the constant delay observer. Observer (dashed-line), System (continuous line).

the corresponding time-varying observer with the same gain $X^{-1}Y$ is able to reconstruct the unmeasured states. We have depicted the system and the observer states in figure 3.

5.2 *Second example*

In this subsection, we show through a case study that the proposed LMI condition (3.2) is not conservative especially when the nonlinearities have large Lipschitz values. To this aim, let us consider the continuous-time delay system described by the following dynamical equations

$$\left.\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 1 & 2 \\ 9 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 1 \end{bmatrix} \\ &\quad \times x(t-d) + \begin{bmatrix} \gamma_1 \dfrac{\sin(x_1(t))}{x_1(t)} \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ \gamma_2(\cos(x_2(t-d)) - 1) \end{bmatrix}, \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t). \end{aligned}\right\} \tag{5.5}$$

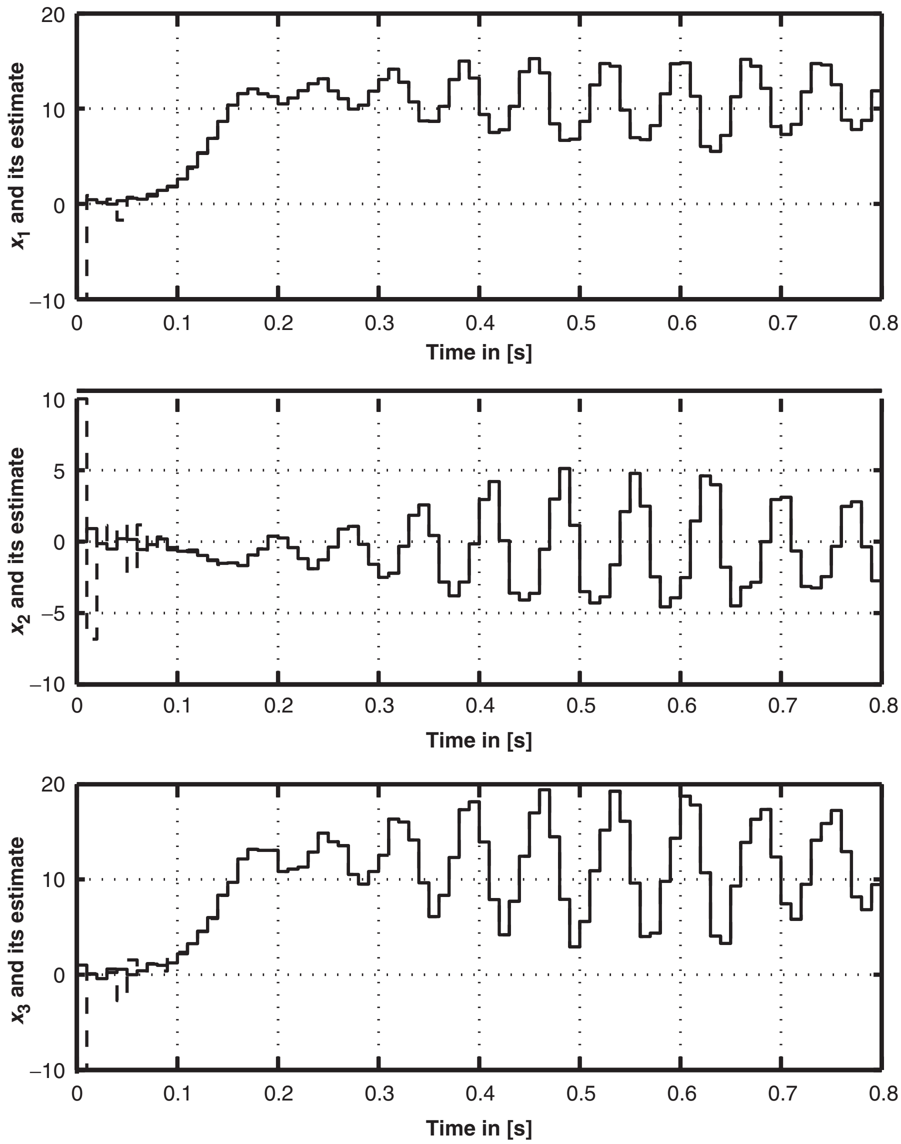

Figure 2. The performance of the time-varying delay observer – Case 1. Observer (dashed-line), System (continuous line).

Here, γ_1 and γ_2 stand for the Lipschitz values of the system nonlinearities. The Euler discretization of (5.5) for $\delta = 0.1$ gives the following states matrices

$$\left.\begin{aligned} &A = \begin{bmatrix} 1.1 & 0.2 \\ 0.9 & 1.3 \end{bmatrix},\ A_d = \begin{bmatrix} 0.05 & 0.02 \\ 0.05 & 0.1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \\ &M_f = \begin{bmatrix} \sqrt{\delta}\gamma_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad N_f = \begin{bmatrix} \sqrt{\delta} & 0 \\ 0 & 0 \end{bmatrix}, \\ &M_g = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\delta}\gamma_2 \end{bmatrix}, \quad N_g = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\delta} \end{bmatrix}. \end{aligned}\right\} \tag{5.6}$$

We have parameterized the matrices M_f and M_g with the positive parameters γ_1 and γ_2 in order to test the solvability of the LMI (3.2) for increasing values of γ_1 and γ_2. In table 1, we have recorded the solution of the LMI (3.2) for $\gamma_1 = 1, 3, 6$ and $\gamma_2 = 1, 3, 5$. For each case, the observer gain exists and the LMI condition (3.2) remains solvable.

6. Conclusion

A convex optimization approach to observation of both constant-delay and time-varying delay nonlinear systems is presented. For both cases, the observation problem

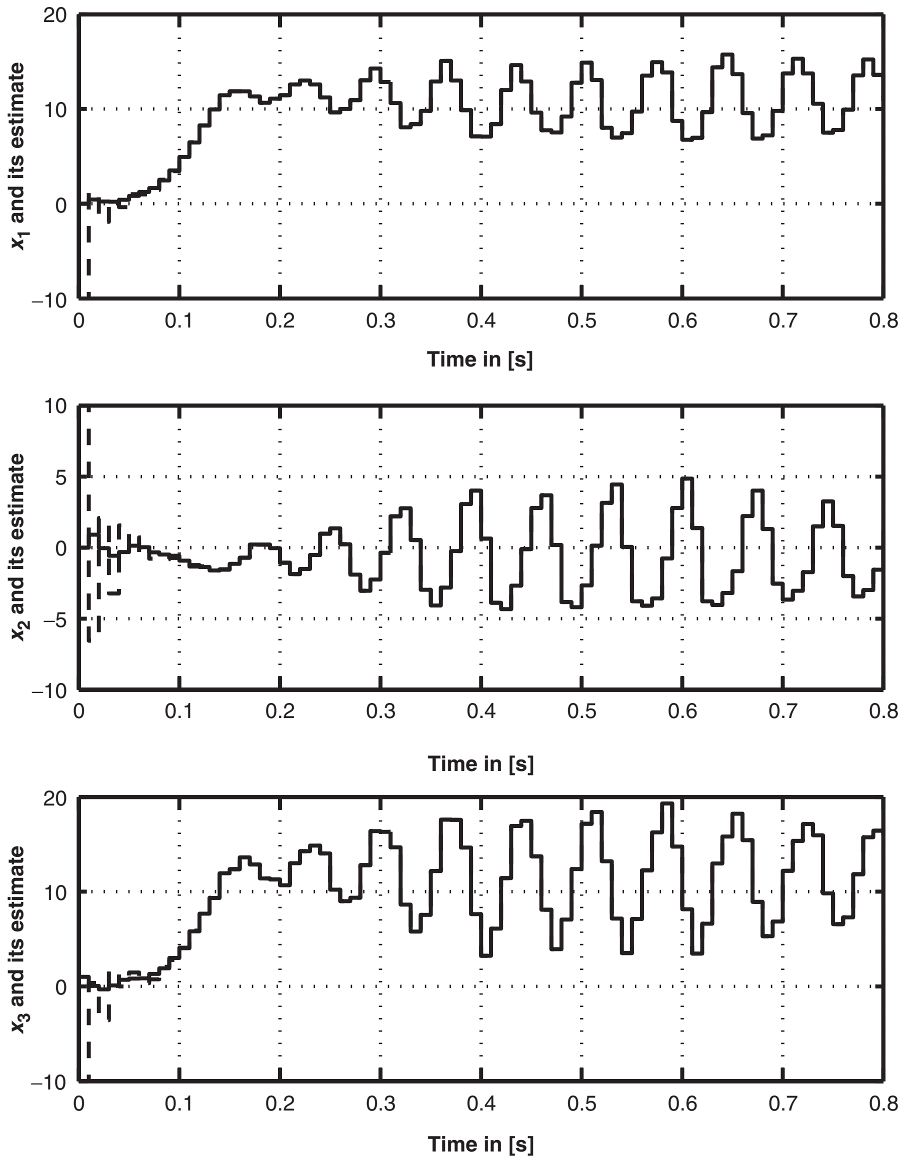

Figure 3. The performance of the time-varying delay observer – Case 2. Observer (dashed-line), System (continuous line).

Table 1. The numerical solution of the LMI (3.2).

γ_1, γ_2	X		Y	Q		ε_1	ε_2
1, 1	0.6420	0.0347	−0.4964	0.2849	0.0795	0.5921	0.6138
	0.0347	0.8642	−0.9479	0.0795	0.4005		
3, 3	0.5583	0.1809	−0.6763	0.2279	0.1844	1.1745	1.7384
	0.1809	1.2135	−1.3703	0.1844	0.5076		
6, 5	0.1108	0.0893	−0.2058	0.0412	0.0656	0.5389	3.0773
	0.0893	0.7774	−0.7947	0.0656	0.4311		

is reduced to a stability problem of linear systems with structured known uncertainties. In case of constant delay systems, the developed linear matrix inequality condition that guarantees the existence of the observer-gain is a delay-independent. For time-varying delay nonlinear systems, we have obtained a delay-dependent condition that is parameterized by the maximum value that can reach the delay function.

For both cases, the delay amount is assumed to be known to set up the dynamics of the nonlinear observer. The presented discrete-time observer can be seen as an extension of discrete-time LMI-based observers that do not involve delay nonlinearities.

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