International Journal of Control
Vol. 78, No. 6, April 2005, 385–395

Observer-based control of discrete-time Lipschitzian non-linear systems: application to one-link flexible joint robot

SALIM IBRIR, WEN FANG XIE* and CHUN-YI SU

Concordia University, Department of Mechanical and Industrial Engineering,
1455 de Maisonneuve Boulvard. West, Montreal, Quebec, Canada H3G 1M8

(*Received 12 November 2004; in final form 1 March 2005*)

The problem of designing asymptotic observers along with observer-based feedbacks for a class of discrete-time non-linear systems is considered. We assume that the system non-linearity is globally Lipschitz and the system is supposed to be stabilizable by a linear controller. Sufficient linear matrix inequality condition is derived to ensure the stability of the considered system under the action of feedback control based on the reconstructed states. A numerical example of a single-link flexible joint robot is presented to illustrate the efficacy of the theoretical developments.

1. Introduction

In most practical situations it can be very expensive, or even impossible, to set up the adequate sensors to measure directly the missing system variables. In those situations, the reconstruction of the unmeasured variables from the knowledge of the system inputs and outputs remains the possible way to achieve the desired objective. This subject has been widely discussed in the linear case where the theory is well investigated and the observability and detectability properties are closely connected to the existence of observers with strong convergence properties. However in the non-linear case, the observer design problem has not a systematic solution and the construction of the non-linear observer is extremely dependent upon the form of non-linearities and the type of inputs that play a fundamental role in determining the system observability.

Besides the difficulties of existence and analysis of asymptotic converging observers for non-linear systems, there are two main problems that arise while a high-gain observer is used in feedback to stabilize the system states. The first problem is how to choose properly the observer gain such that the non-linear observer, with proportional injection term, can reproduce the states of the system being observed. This task, generally encountered in high-gain observer design, is known to be a challenging issue, and solution to this fundamental problem may fail when non-linearities are of high Lipschitz constants. For control and observation designs, the global Lipschitz property is a rather restrictive condition, but it can be made satisfied in well-defined state space region. Successfully practical examples describing this class of systems are numerous, see for example Huijberts *et al*. (2001), Liao and Haung (1999), Pogromsky and Nijmeijer (1998) and Song and Grizzle (1995). For more details on hybrid control and estimation of piecewise affine systems, the reader is referred to the following references (Ferrari-Trecate 2002, Cuzzola and Morari 2002, Balluchi *et al*. 2002, Pettersson and Lennartson 2002 and Li *et al*. 2003).

In our opinion, the failure of designing an appropriate high-gain observer-based feedback is fundamentally due to the manner how one elaborates stability conditions under partial state measurements. Major approximations and simplification lead generally to restrictive conditions. In addition, if the system admits several modes or configurations, the problem of designing a constant-gain observer for the different system modes with different Lipschitz constants becomes more and more complicated, see e.g., Balluchi *et al*. (2001) and Ferrari-Trecate *et al*. (2002). Consequently, some methodological issues arise and call for new look at

*Corresponding author. Email: wfxie@me.concordia.ca

International Journal of Control
ISSN 0020–7179 print/ISSN 1366–5820 online © 2005 Taylor & Francis Group Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170500096666

the proposed approaches. The second major problem is the stability of the observer-based controller, since the construction of observers and controllers independently does not imply that the observer-controllers will ensure systematically the whole system stability.

Observer design for discrete-time non-linear systems has been the subject of numerous research papers, see e.g., Ciccarella *et al.* (1993), Lee and Nam (1991), Moraal and Grozzle (1995), Reif *et al.* (1999) and Song and Grizzle (1995). However, a little attention has been devoted to observer-based control in the discrete-time case. The majority of works that dealt with observer-based control were developed in the continuous-time case for special classes of uncertain linear systems (Lien 2004), and non-linear systems that verify certain growth conditions. A separation principle for a class of non-linear systems has been given in Atassi and Khalil (1999) and Dabroom and Khalil (2001) where semiglobal stability is guaranteed by means of dynamic output feedback. In this paper a linear matrix inequality approach is proposed for both robust observer design and observer-based control of a class of discrete-time non-linear systems. First, a high-gain Luenberger observer is proposed and condition of the existence of such observer gain is given in term of a less restrictive efficient linear matrix inequality. It is worthwhile to mention that the proposed LMI condition is easily derived without any major approximation or additional restrictive conditions. This leads to a quite simple condition that is numerically tractable with any LMI commercial software. Second, the condition of the system stability under the action of observer-based controller is given in term of efficient linear matrix inequality. The one-link flexible joint robot is used as an example to show the usefulness of the developed results.

Throughout this paper we note by $\mathbb{R}$, $\mathbf{0}$ and $\boldsymbol{I}$ the set of real numbers, the null matrix, and the identity matrix of appropriate dimensions, respectively. $\|\cdot\|$ stands for the habitual Euclidean norm. The notation $A>0$ (resp. $A<0$) means that the matrix A is positive definite (resp. negative definite). A' is the matrix transpose of A. "$\star$" is used to notify an element which is induced by transposition. At first, we recall some basic lemmas that are frequently used in setting the proofs of the paper statements.

Lemma 1 (The Schur complement lemma) (Boyd *et al.* 1994)**:** *Given constant matrices M, N, Q of appropriate dimensions where M and Q are symmetric, then $Q>0$ and $M+N'Q^{-1}N<0$ if and only if*

$$\begin{bmatrix} M & N' \\ N & -Q \end{bmatrix} < 0,$$

or equivalently

$$\begin{bmatrix} -Q & N \\ N' & M \end{bmatrix} < 0.$$

Lemma 2 (Gu 2000)**:** *For any constant symmetric matrix $M\in\mathbb{R}^{n\times n}$, $M=M'>0$, scalar $\gamma>0$, vector function ω: $[0,\gamma]\mapsto\mathbb{R}^n$ such that the integration in the following is well defined, we have*

$$\gamma\int_0^\gamma \omega'(\beta)M\omega(\beta)\,\mathrm{d}\beta \geq \left(\int_0^\gamma \omega(\beta)\,\mathrm{d}\beta\right)' M\left(\int_0^\gamma \omega(\beta)\,\mathrm{d}\beta\right). \quad (1)$$

2. Observer design

Observer design for non-linear discrete-time systems has been considered in many research papers, see e.g., Ciccarella *et al.* (1993), Lee and Nam (1991), Reif *et al.* (1999) and Song and Grizzle (1995). In this section we consider the problem of observer design for a class of discrete-time non-linear systems where the asymptotic convergence of the estimates is guaranteed without any approximation of the non-linear terms of the system being observed. The convergence of the estimates is conditioned by the solution of a less restrictive linear matrix inequality. Consider the discrete-time non-linear system

$$\begin{cases} x_{k+1} = Ax_k + f(x_k) + Bu_k \\ y_k = Cx_k, \end{cases} \quad (2)$$

where the pair (A, C) is assumed to be detectable, $x_k\in\mathscr{M}\subset\mathbb{R}^n$, $u_k\in\mathscr{U}$ is an m dimensional control input, $\mathscr{U}$ is the set of bounded inputs for which system (2) is observable. $y_k\in\mathbb{R}^p$ is the system output, and $f:\mathscr{M}\to\mathbb{R}^n$ is a Lipschitz non-linearity satisfying $f(0)=0$ and

$$\frac{\partial f}{\partial x_k}(x_k) = \mathscr{G}(x_k) = M\,F(x_k)\,N, \quad (3)$$

where $M\in\mathbb{R}^{n\times n}$, $N\in\mathbb{R}^{n\times n}$ are well-defined real matrices, and $F(x_k)\in\mathbb{R}^{n\times n}$ is a norm-bounded matrix satisfying $F'(x_k)F(x_k)\leq I$.

Remark 1: The Lipschitz property as formulated in (3) does not involve any approximation of non-linearities by their norms. Therefore, this important formulation will reduce the conservatism of the results and makes the design of the non-linear observer dependent on the non-linearities as they appear.

We propose an observer of the form

$$\hat{x}_{k+1} = A\hat{x}_k + f(\hat{x}_k) + Bu_k + P^{-1}Y(C\hat{x}_k - y_k), \quad (4)$$

where $P = P' > 0$ is an $n \times n$ matrix and Y is an arbitrary matrix of dimension $n \times p$. If we define $e_k = \hat{x}_k - x_k$ as the observation error, then we obtain

$$\begin{aligned} e_{k+1} &= \big(A + P^{-1}YC\big)e_k + f(\hat{x}_k) - f(x_k) \\ &= \big(A + P^{-1}YC\big)e_k + \int_0^1 MF(s_k)|_{s_k=\hat{x}_k-\lambda e_k} N\, e_k\, \mathrm{d}\lambda \\ &= \int_0^1 \big(A + P^{-1}YC + MF(s_k)|_{s_k=\hat{x}_k-\lambda e_k} N\big)e_k\, \mathrm{d}\lambda. \qquad (5) \end{aligned}$$

Setting $V_k = e_k' P e_k$ as a Lyapunov function candidate, then we have

$$\begin{aligned} &V_{k+1} - V_k \\ &= \left[\int_0^1 e_k'\big(A' + C'Y'P^{-1} + N'F'(s_k)|_{s_k=\hat{x}_k-\lambda e_k} M'\big)\, \mathrm{d}\lambda\right] P \\ &\quad \times \left[\int_0^1 \big(A + P^{-1}YC + MF(s_k)|_{s_k=\hat{x}_k-\lambda e_k} N\big)e_k\right] \\ &\quad - \int_0^1 e_k' P e_k\, \mathrm{d}\lambda. \qquad (6) \end{aligned}$$

Using the result of Lemma 2, we obtain

$$\begin{aligned} V_{k+1} - V_k \le \int_0^1 e_k'\big[\big(A' + Y'P^{-1} + N'F'(s_k)|_{s_k=\hat{x}_k-\lambda e_k} M'\big)P \\ \times \big(A + P^{-1}Y + MF(s_k)|_{s_k=\hat{x}_k-\lambda e_k} N\big) - P\big]e_k\, \mathrm{d}\lambda. \end{aligned} \qquad (7)$$

Using the Schur complement lemma, we can obtain $V_{k+1} - V_k < 0$ if the following inequality is satisfied

$$\begin{bmatrix} -P & A'P + C'Y' + \int_0^1 N'F'(s_k)|_{s_k=\hat{x}_k-\lambda e_k} M'P\, \mathrm{d}\lambda \\ \star & -P \end{bmatrix} < 0. \qquad (8)$$

The last inequality is rewritten as follows

$$\begin{aligned} &\begin{bmatrix} -P & A'P + C'Y' \\ \star & -P \end{bmatrix} \\ &\quad + \int_0^1 \begin{bmatrix} \mathbf{0} \\ PM \end{bmatrix} F(s_k)|_{s_k=\hat{x}_k-\lambda e_k} \begin{bmatrix} N' & \mathbf{0} \end{bmatrix} \mathrm{d}\lambda \\ &\quad + \int_0^1 \begin{bmatrix} N' \\ 0 \end{bmatrix} F'(s_k)|_{s_k=\hat{x}_k-\lambda e_k} \begin{bmatrix} \mathbf{0} & M'P \end{bmatrix} \mathrm{d}\lambda < 0. \end{aligned}$$

Since for any $\epsilon > 0$

$$\begin{aligned} &\int_0^1 \begin{bmatrix} \mathbf{0} \\ PM \end{bmatrix} F(s_k)|_{s_k=\hat{x}_k-\lambda e_k} \begin{bmatrix} N' & \mathbf{0} \end{bmatrix} \mathrm{d}\lambda \\ &\quad + \int_0^1 \begin{bmatrix} N' \\ \mathbf{0} \end{bmatrix} F'(s_k)|_{s_k=\hat{x}_k-\lambda e_k} \begin{bmatrix} \mathbf{0} & M'P \end{bmatrix} \mathrm{d}\lambda \\ &\le \frac{1}{\epsilon} \begin{bmatrix} \mathbf{0} \\ PM \end{bmatrix} \begin{bmatrix} \mathbf{0} & M'P \end{bmatrix}. \\ &\quad + \epsilon \begin{bmatrix} N' \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} N & \mathbf{0} \end{bmatrix}. \end{aligned}$$

This implies that if the following linear matrix inequality

$$\begin{bmatrix} -P + \epsilon N'N & A'P + C'Y' \\ \star & -P + \frac{1}{\epsilon} PMM'P \end{bmatrix} < 0, \qquad (9)$$

then $V_{k+1} - V_k < 0$. Inequality (9) is equivalent by the Schur complement to the following LMI

$$\begin{bmatrix} -P + \epsilon N'N & A'P + C'Y' & \mathbf{0} \\ \star & -P & PM \\ \star & \star & -\epsilon \boldsymbol{I} \end{bmatrix} < 0. \qquad (10)$$

We summarize the result in the following statement.

Theorem 1: *Consider system* (2). *If there exist a symmetric and positive definite matrix* $P \in \mathbb{R}^{n\times n}$, *a matrix* $Y \in \mathbb{R}^{n\times p}$, *and a positive constant* ϵ *such that the linear matrix inequality* (10) *holds. Then the states of observer* (4) *converge asymptotically to the states of system* (2) *when time elapses.*

As a direct consequence, the linear matrix inequality (10) can be exploited to design of switching observers for switched systems of the form

$$\begin{aligned} x_{k+1}^{(i)} &= A_i\, x_k^{(i)} + f_i(x_k^{(i)}) + B\, u_k^{(i)}, \\ y_k^{(i)} &= C_i\, x_k^{(i)}, \end{aligned} \qquad (11)$$

where $i \in \mathscr{S} = \{1, \ldots, s\}$ is the current mode of system (11), s is the number of the system modes, and $f_i(x_k^{(i)})$ is a Lipschitz non-linear term that verifies for each mode i

$$\begin{aligned} &\frac{\partial f_i}{\partial x_k^{(i)}}(x_k^{(i)}) = \mathscr{G}_i(x_k) \\ &\quad = M_i F_i(x_k^{(i)}) N_i, \quad i \in \mathscr{S}, \quad F'_i(x_k^{(i)}) F_i(x_k^{(i)}) \le \boldsymbol{I}. \qquad (12) \end{aligned}$$

In this case, the solution of the s linear matrix inequalities

$$\begin{bmatrix} -P_i + \epsilon_i N_i' N_i & A_i' P_i + C_i' Y_i' & \mathbf{0} \\ \star & -P_i & P_i M_i \\ \star & \star & -\epsilon_i \boldsymbol{I} \end{bmatrix} < 0, \quad i \in \mathscr{S} \qquad (13)$$

with respect to the matrices $(P_i)_{1\le i\le s}$, $(Y_i)_{1\le i\le s}$, $(\epsilon_i)_{1\le i\le s}$, gives the current switching observer gain $L_i = P_i^{-1} Y_i$.

3. Observer-based control

Since the observation error of a Lipschitz non-linear system can be stabilized by static output feedback, one can also investigate the possibility of stabilizing a Lipschitz non-linear system by a full static feedback. In this section, we shall study and derive sufficient conditions under which a discrete-time Lipschitz non-linear system is globally asymptotically stable under the action of observer-based linear static feedback. The main objective is to conceive an observer of the following form

$$\hat{x}_{k+1} = A\,\hat{x}_k + f(\hat{x}_k) + Bu_k + L(C\hat{x}_k - y_k), \tag{14}$$

such that the closed-loop system (2) under the feedback $u_k = K\hat{x}_k$ is globally asymptotically stable. Let $e_k = \hat{x}_k - x_k$ be the observation error then, we write

$$\left.\begin{aligned} x_{k+1} &= (A+BK)x_k + \int_0^1 \mathscr{G}(s)\big|_{s=(1-\lambda)x_k} x_k \,\mathrm{d}\lambda + BKe_k,\\ e_{k+1} &= (A+LC)e_k + \int_0^1 \mathscr{G}(s)\big|_{s=\hat{x}_k-\lambda e_k} e_k \,\mathrm{d}\lambda. \end{aligned}\right\} \tag{15}$$

Let

$$\Phi(x_k, \hat{x}_k, \lambda) = \begin{bmatrix} A+BK+\mathscr{G}(s)\big|_{s=(1-\lambda)x_k} & BK \\ \mathbf{0} & A+LC+\mathscr{G}(s)\big|_{s=\hat{x}_k-\lambda e_k} \end{bmatrix}. \tag{16}$$

Then

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \int_0^1 \Phi(x_k, \hat{x}_k, \lambda)\begin{bmatrix} x_k \\ e_k \end{bmatrix}\mathrm{d}\lambda. \tag{17}$$

Let $Z = Z' > 0$ be a matrix of dimension $2n \times 2n$. Then system (17) is asymptotically stable if there exists a Lypunov function

$$W_k = \begin{bmatrix} x_k \\ e_k \end{bmatrix}' Z \begin{bmatrix} x_k \\ e_k \end{bmatrix}, \tag{18}$$

such that $W_{k+1} - W_k < 0$. We have

$$\begin{aligned} W_{k+1} - W_k = {}& \left(\int_0^1 \begin{bmatrix} x_k \\ e_k \end{bmatrix}' \Phi'(x_k, \hat{x}_k, \lambda)\,\mathrm{d}\lambda\right) \\ & \times Z\left(\int_0^1 \Phi(x_k, \hat{x}_k, \lambda)\begin{bmatrix} x_k \\ e_k \end{bmatrix}\mathrm{d}\lambda\right) \\ & - \begin{bmatrix} x_k \\ e_k \end{bmatrix} Z \begin{bmatrix} x_k \\ e_k \end{bmatrix}. \end{aligned} \tag{19}$$

Using result of Lemma 2, we can write that

$$\begin{aligned} W_{k+1} - W_k \le {}& \int_0^1 \left(\begin{bmatrix} x_k \\ e_k \end{bmatrix}' \Phi'(x_k, \hat{x}_k, \lambda)\right) \\ & \times Z\left(\Phi(x_k, \hat{x}_k, \lambda)\begin{bmatrix} x_k \\ e_k \end{bmatrix}\right)\mathrm{d}\lambda \\ & - \int_0^1 \begin{bmatrix} x_k \\ e_k \end{bmatrix}' Z \begin{bmatrix} x_k \\ e_k \end{bmatrix}\mathrm{d}\lambda. \end{aligned} \tag{20}$$

This implies that $W_{k+1} - W_k \le 0$ if

$$\int_0^1 \Phi'(x_k, \hat{x}_k, \lambda) Z \Phi(x_k, \hat{x}_k, \lambda) - Z \,\mathrm{d}\lambda < 0 \tag{21}$$

or equivalently

$$\int_0^1 \begin{bmatrix} -Z & \Phi'(x_k, \hat{x}_k, \lambda)Z \\ Z\Phi(x_k, \hat{x}_k, \lambda) & -Z \end{bmatrix}\mathrm{d}\lambda < 0. \tag{22}$$

Let

$$Z = \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix}, \quad K = \bar{Z}_1^{-1}\widetilde{K}, \quad L = Z_2^{-1} Y, \quad Z_1 B = B\bar{Z}_1, \tag{23}$$

where Z_1 and Z_2 are symmetric and positive definite matrices of dimensions $n \times n$ and $\bar{Z}_1$ is a full rank matrix to be determined with the new vectors $\widetilde{K}$ and Y of appropriate dimensions. By replacing all these new variables in inequality (22), we obtain the new condition of stability

$$\begin{aligned} & \begin{bmatrix} -Z_1 & \mathbf{0} & A'Z_1 + \widetilde{K}'B' & \mathbf{0} \\ \mathbf{0} & -Z_2 & \widetilde{K}'B' & A'Z_2 + C'Y' \\ \star & \star & -Z_1 & \mathbf{0} \\ \star & \star & \star & -Z_2 \end{bmatrix} \\ & + \int_0^1 \begin{bmatrix} \mathbf{0} & \mathbf{0} & N'F'(s)\big|_{s=(1-\lambda)x_k} M'Z_1 & \mathbf{0} \\ \star & \mathbf{0} & \mathbf{0} & N'F'(s)\big|_{s=\hat{x}_k-\lambda e_k} M'Z_2 \\ \star & \star & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \mathbf{0} \end{bmatrix}\mathrm{d}\lambda < 0. \end{aligned} \tag{24}$$

We have

$$
\begin{aligned}
&\int_0^1 \begin{bmatrix} 0 & \mathbf{0} & N'F'(s)|_{s=(1-\lambda)x_k} M'Z_1 & \mathbf{0} \\ \star & \mathbf{0} & \mathbf{0} & N'F'(s)|_{s=\hat{x}_k-\lambda e_k} M'Z_2 \\ \star & \star & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \mathbf{0} \end{bmatrix} d\lambda \\
&= \int_0^1 \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ Z_1 M \\ \mathbf{0} \end{bmatrix} F(s)|_{s=(1-\lambda)x_k} \begin{bmatrix} N & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} d\lambda \\
&\quad + \int_0^1 \begin{bmatrix} N & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}' F'(s)|_{s=(1-\lambda)x_k} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ Z_1 M \\ \mathbf{0} \end{bmatrix}' d\lambda \\
&\quad + \int_0^1 \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ Z_2 M \end{bmatrix} F(s)|_{s=\hat{x}_k-\lambda e_k} \begin{bmatrix} \mathbf{0} & N & \mathbf{0} & \mathbf{0} \end{bmatrix} d\lambda \\
&\quad + \int_0^1 \begin{bmatrix} \mathbf{0} & N & \mathbf{0} & \mathbf{0} \end{bmatrix}' F'(s)|_{s=\hat{x}_k-\lambda e_k} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ Z_2 M \end{bmatrix}' d\lambda \\
&\le \mu_1 \begin{bmatrix} N & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}' \begin{bmatrix} N & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mu_1^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ Z_1 M \\ \mathbf{0} \end{bmatrix}' \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ Z_1 M \\ \mathbf{0} \end{bmatrix} \\
&\quad + \mu_2 \begin{bmatrix} \mathbf{0} & N & \mathbf{0} & \mathbf{0} \end{bmatrix}' \begin{bmatrix} \mathbf{0} & N & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mu_2^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ Z_2 M \end{bmatrix}' \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ Z_2 M \end{bmatrix} \\
&= \begin{bmatrix} \mu_1 N'N & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \mu_2 N'N & \mathbf{0} & \mathbf{0} \\ \star & \star & \mu_1^{-1} Z_1 M M' Z_1 & \mathbf{0} \\ \star & \star & \star & \mu_2^{-1} Z_2 M M' Z_2 \end{bmatrix}.
\end{aligned}
$$

Then from (24), we conclude that $W_{k+1} - W_k < 0$ if the following LMI holds

$$
\begin{bmatrix} -Z_1 + \mu_1 N'N & \mathbf{0} & A'Z_1 + \widetilde{K}'B' & \mathbf{0} \\ \star & -Z_2 + \mu_2 N'N & \widetilde{K}'B' & A'Z_2 + C'Y' \\ \star & \star & -Z_1 + \mu_1^{-1} Z_1 M M' Z_1 & \mathbf{0} \\ \star & \star & \star & -Z_2 + \mu_2^{-1} Z_2 M M' Z_2 \end{bmatrix} < 0 \tag{25}
$$

which is equivalent by the Schur complement lemma to the following LMI

$$
\begin{bmatrix} -Z_1 + \mu_1 N'N & \mathbf{0} & A'Z_1 + \widetilde{K}'B' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & -Z_2 + \mu_2 N'N & \widetilde{K}'B' & A'Z_2 + C'Y' & \mathbf{0} & \mathbf{0} \\ \star & \star & -Z_1 & \mathbf{0} & Z_1 M & \mathbf{0} \\ \star & \star & \star & -Z_2 & \mathbf{0} & Z_2 M \\ \star & \star & \star & \star & -\mu_1 \boldsymbol{I} & \mathbf{0} \\ \star & \star & \star & \star & \star & -\mu_2 \boldsymbol{I} \end{bmatrix} < 0. \tag{26}
$$

We summarize the result in the following statement.

Theorem 2: *Consider system* (2) *and observer* (14) *under the feedback* $u_k = \bar{Z}_1^{-1}\widetilde{K}\hat{x}_k$ *with* $L = Z_2^{-1}Y$. *Then if there exist two symmetric and positive definite matrices* Z_1 *and* Z_2 *of dimensions* $n \times n$, *a matrix* $\bar{Z}_1 \in \mathbb{R}^{m\times m}$, *a vector* $\widetilde{K} \in \mathbb{R}^{m\times n}$, *a matrix* $Y \in \mathbb{R}^{n\times p}$, *and two positive constants* μ_1 *and* μ_2 *such that the linear matrix* (26) *holds under the constraint* $Z_1 B = B\bar{Z}_1$. *Then system* (2) *is globally asymptotically stable under the action of the observer-based controller* $u_k = \bar{Z}_1^{-1}\widetilde{K}\hat{x}_k$.

Due to the equality constraint given in theorem 2, the search of the observer and the controller gains may be difficult or restrictive for certain non-linear systems with single input. This is due to the fact that the constraint $Z_1 B = B\bar{Z}_1$ is reduced to $Z_1 B = cB$ where c is a real constant. Otherwise, the solvability of the constraint $Z_1 B = B\bar{Z}_1$ depends on the positive definiteness of the matrix Z_1 and the column rank of the matrix B. If $Z_1 > 0$ and B has a full column rank, then $\bar{Z}_1$ is invertible.

If the LMI (26) is not solvable or in order to avoid solving (26) under the constraint $Z_1 B = B\bar{Z}_1$, we proceed with two independent steps. First, we try to find a vector gain K such that system

$$
x_{k+1} = (A + BK)x_k + f(x_k), \tag{27}
$$

is globally asymptotically stable. Let $\mathscr{A} = A + BK$, then starting from inequality (21), we write

$$
\int_0^1 \begin{bmatrix} -S & \Phi'(x_k, \hat{x}_k, \lambda)S \\ S\Phi(x_k, \hat{x}_k, \lambda) & -S \end{bmatrix} d\lambda < 0. \tag{28}
$$

Let us put $L = S_2^{-1}Y_2$ where $Y_2 \in \mathbb{R}^{n\times p}$ and $S_1 \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix such that for a given symmetric and positive definite matrix $S_2 \in \mathbb{R}^{n\times n}$ the matrix S is defined as

$$S = \begin{bmatrix} S_1 & \mathbf{0} \\ \mathbf{0} & S_2 \end{bmatrix}.$$

The matrix inequality condition (28) leads to the following inequality by replacing $A'Z_1 + \widetilde{K}'B'$ by the matrix $A_k'S_1$ and Y by Y_2 in inequality (25). That is

$$\begin{bmatrix} -S_1 + \mu_1 N'N & \mathbf{0} & \mathscr{A}'S_1 & \mathbf{0} \\ \star & -S_2 + \mu_2 N'N & K'B'S_1 & A'S_2 + C'Y_2' \\ \star & \star & -S_1 + \mu_1^{-1} S_1 M M' S_1 & \mathbf{0} \\ \star & \star & \star & -S_2 + \mu_2^{-1} S_2 M M' S_2 \end{bmatrix} < 0, \tag{29}$$

or equivalently

$$\begin{bmatrix} -S_1 + \mu_1 N'N & \mathbf{0} & \mathscr{A}'S_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & -S_2 + \mu_2 N'N & K'B'S_1 & A'S_2 + C'Y_2' & \mathbf{0} & \mathbf{0} \\ \star & \star & -S_1 & \mathbf{0} & S_1 M & \mathbf{0} \\ \star & \star & \star & -S_2 & \mathbf{0} & S_2 M \\ \star & \star & \star & \star & -\mu_1 \boldsymbol{I} & \mathbf{0} \\ \star & \star & \star & \star & \star & -\mu_2 \boldsymbol{I} \end{bmatrix} < 0. \tag{30}$$

It remains now to study the stability of system (27). Let us put $K = Y_1 X^{-1}$ where $X \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix, and $Y_1 \in \mathbb{R}^{m\times n}$ is some constant matrix to be determined. By choosing $\mathscr{V}_k = x_k' X^{-1} x_k$ as a Lyapunov function to the dynamics (27), and taking into account that $f(0) = 0$, we write

$$x_{k+1} = (A + BY_1X^{-1})x_k + \int_0^1 MF(s_k)|_{s_k=(1-\lambda)x_k} N x_k \, \mathrm{d}\lambda. \tag{31}$$

System (31) is in form of (5). Then by the use of result of theorem 1, we conclude that $\mathscr{V}_{k+1} - \mathscr{V}_k < 0$ if the following hold

$$\int_0^1 \Big\{ \big(A + BY_1X^{-1} + MF(\cdot)N\big)' X^{-1} \times \big(A + BY_1X^{-1} + MF(\cdot)N\big) - X^{-1} \Big\} \, \mathrm{d}\lambda < 0.$$

Pre- and post multiplying the last inequality by X, we obtain the new condition of stability

$$\int_0^1 \Big\{ X\big(A + BY_1X^{-1} + MF(\cdot)N\big)' X^{-1} \times \big(A + BY_1X^{-1} + MF(\cdot)N\big) X - X \Big\} \, \mathrm{d}\lambda < 0,$$

which is equivalent by the Schur complement to the following LMI

$$\begin{bmatrix} -X & XA' + Y_1'B' + \int_0^1 XN'F'(\cdot)M' \, \mathrm{d}\lambda \\ \star & -X \end{bmatrix} < 0.$$

The last inequality is verified if there exists $\tau > 0$ such that

$$\begin{bmatrix} -X & XA' + Y_1'B' \\ \star & -X \end{bmatrix} + \frac{1}{\tau} \begin{bmatrix} XN' \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} NX & \mathbf{0} \end{bmatrix} + \tau \begin{bmatrix} \mathbf{0} \\ M \end{bmatrix} \begin{bmatrix} \mathbf{0} & M' \end{bmatrix} < 0.$$

Finally, we get the sufficient LMI condition that guarantees the stability of system (31) by the linear feedback $u_k = Kx_k$, that is

$$\begin{bmatrix} -X & XA' + Y_1'B' & XN' \\ \star & -X + \tau MM' & \mathbf{0} \\ \star & \star & -\tau \boldsymbol{I} \end{bmatrix} < 0. \tag{32}$$

We summarize the design of the observer-controller gain in the following algorithm.

Algorithm 1: *The design of the observer and the controller gains is given in two steps.*

Step 1. *Solve the LMI* (32) *with respect to* X, Y_1, *and* τ *and set* $K = Y_1 X^{-1}$ *if a solution exists.*
Step 2. *For the value of* K *given in step* 1, *solve the LMI* (30) *with respect to* S_1 *and* S_2, μ_1, μ_2, *and* Y_2. *If a solution exist the observer gain is set to* $L = S_2^{-1} Y_2$.

In summary, the controller gain is computed in step 1 of the last algorithm and in step 2, the observer gain is calculated in accordance to the fixed controller gain K so as the observer-controller ensures the stability of the non-linear system. The main advantage of this algorithm is that the equality constraint of Theorem 2 is omitted. But in the same time two linear matrix inequalities are required to achieve the computation of the observer-controller gains.

Extension of the observer-based LMIs conditions for multiple-models non-linear discrete-time systems is also feasible. To deliver such conditions, it is sufficient to replace in the developed LMIs all the matrices by their corresponding current mode matrices. One can also consider the problem of constant-gain observer-based control that does not require any mode detection.

4. Application

Let us consider the single-link flexible joint robot described by the following dynamics (Raghavan and Hedrick 1994)

$$\left.\begin{aligned} \dot\theta_m &= \omega_m,\\ \dot\omega_m &= \frac{k}{J_m}(\theta_\ell - \theta_m) - \frac{B}{J_m}\omega_m + \frac{K_\tau}{J_m}u,\\ \dot\theta_\ell &= \omega_\ell,\\ \dot\omega_\ell &= -\frac{k}{J_\ell}(\theta_\ell - \theta_m) - \frac{mgh}{J_\ell}\sin(\theta_\ell), \end{aligned}\right\} \quad (33)$$

where J_m represents the inertia of the actuator (d.c. motor), and J_ℓ stands for the inertia of the link. θ_m and θ_ℓ are the angles of rotations of the motor and the link, respectively. $\dot\theta_m$ and $\dot\theta_\ell$ are their angular velocities. k, K_τ, m, g, and h are positive constants, see table 1. For the parameters given in table 1, we can write system (33) as follows

$$\left.\begin{aligned} \dot x &= F\,x + g(x) + D\,u,\\ y &= C\,x, \end{aligned}\right\} \quad (34)$$

Table 1. Robot parameters.

System parameters	Values
Motor inertia, J_m (kg m^2)	3.7×10^{-3}
Link inertia, J_ℓ (kg m^2)	9.3×10^{-3}
Pointer mass, m (kg)	2.1×10^{-1}
Link length, $2b$ (m)	3.1×10^{-1}
Torsional spring constant, k (Nm rad^{-1})	1.8×10^{-1}
Viscous friction coefficient, B (Nm V^{-1})	4.6×10^{-2}
Amplifier gain, K_τ (Nm V^{-1})	8×10^{-2}

where

$$F = \begin{bmatrix} 0 & 1 & 0 & 0\\ -48.6 & -1.25 & 48.6 & 0\\ 0 & 0 & 0 & 1\\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & \mathbf{1} & 0 & 0 \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 0\\ 0\\ 0\\ -3.33\sin(x_3) \end{bmatrix}, \quad D = \begin{bmatrix} 0\\ 21.6\\ 0\\ 0 \end{bmatrix}.$$

The Euler discretization of the dynamic system (34) gives

$$\left.\begin{aligned} x_{k+1} &= A\,x_k + f(x_k) + B\,u_k,\\ y_k &= C\,x_k, \end{aligned}\right\} \quad (35)$$

where $A = I + \delta F$, $f(x_k) = \delta\,g(x_k)$, $B = \delta\,D$, where δ is the sampling period. The Jacobian of the non-linearity can be written as

$$\mathcal{G}(x_k) = \delta\frac{\partial g(x_k)}{\partial x_k} = M\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \cos(x_3(k)) & 0\\ 0 & 0 & 0 & 0 \end{bmatrix} N, \quad (36)$$

where

$$N = \delta\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \sqrt{3.33} & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \sqrt{3.33} & 0 \end{bmatrix}.$$

In order to solve the presented LMIs, we used the package of LMIs of MATLAB. Solution of the linear matrix inequality (10) with respect to P, Y, and ϵ gives

$$P = \begin{bmatrix} 0.8478 & 0 & 0 & 0 \\ 0 & 0.8910 & -0.4535 & 0.0069 \\ 0 & -0.4535 & 1.0912 & -0.0763 \\ 0 & 0.0069 & -0.0763 & 0.0203 \end{bmatrix},$$

$$Y = \begin{bmatrix} -0.8478 & -0.0085 \\ 0.4317 & -0.8473 \\ -0.2055 & 0.0798 \\ -0.0006 & 0.0025 \end{bmatrix}, \quad \epsilon = 1.0904,$$

$$L = \begin{bmatrix} -1 & -0.0100 \\ 0.4860 & -1.2108 \\ 0 & -0.5328 \\ -0.1950 & -1.4674 \end{bmatrix}.$$

The performance of the observer is represented in figures 1–2 where the input $u_k = 0.1(V)$.

In order to determine the gains of the observer-based controller for system (35), we shall follow the steps of Algorithm 1. For $\delta = 0.01[S]$, and

$$N = \sqrt{\delta}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M = \sqrt{\delta}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3.33 & 0 \end{bmatrix},$$

the solution of the linear matrix inequality (32) with respect to their variables is

$$X = \begin{bmatrix} 0.2671 & -0.1624 & 0.2571 & -0.3757 \\ -0.1624 & 1.4421 & 0.0477 & 0.1020 \\ 0.2571 & 0.0477 & 0.3432 & -0.2086 \\ -0.3757 & 0.1020 & -0.2086 & 2.0077 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0.0061 & -4.9997 & -0.1952 & -0.2121 \end{bmatrix},$$
$$\tau = 0.4785.$$

As a second step, we use the gain K obtained in step 1 to solve the linear matrix inequality (30) with respect

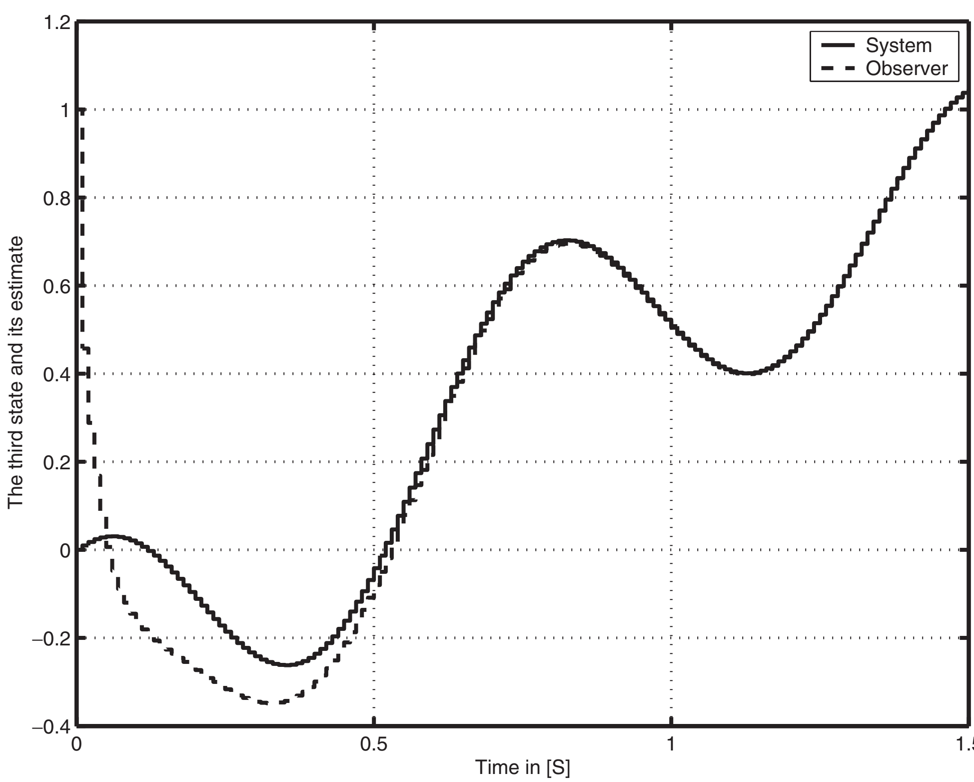

Figure 1. The state $x_3(k)$ and its estimate $\hat{x}_3(k)$.

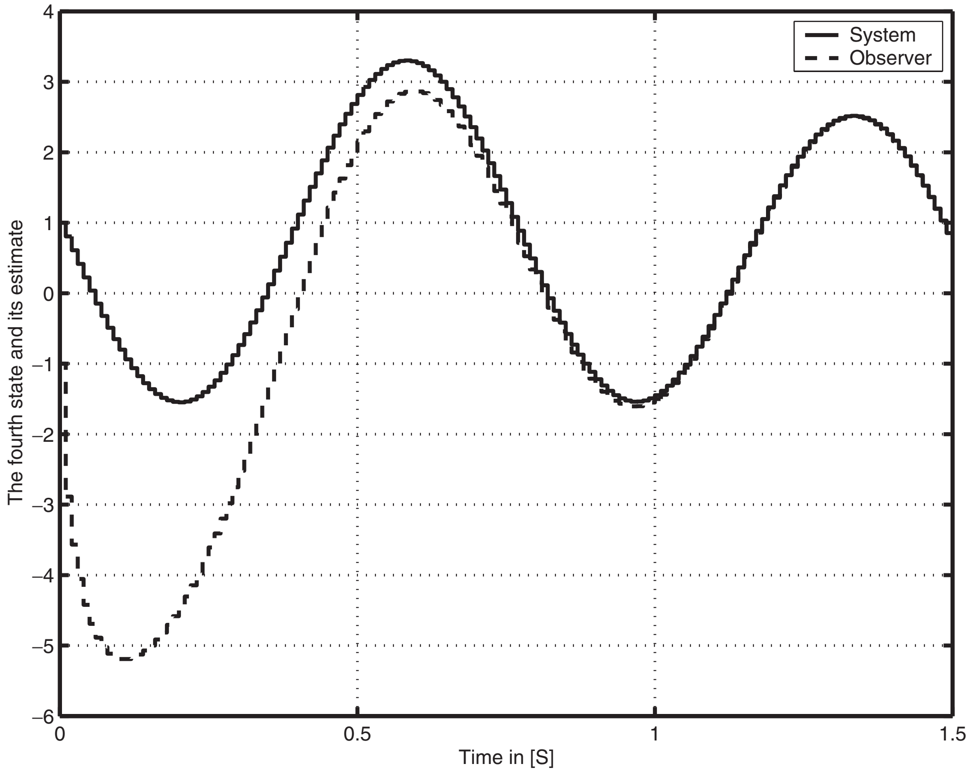

Figure 2. The state $x_4(k)$ and its estimate $\hat{x}_k(4)$.

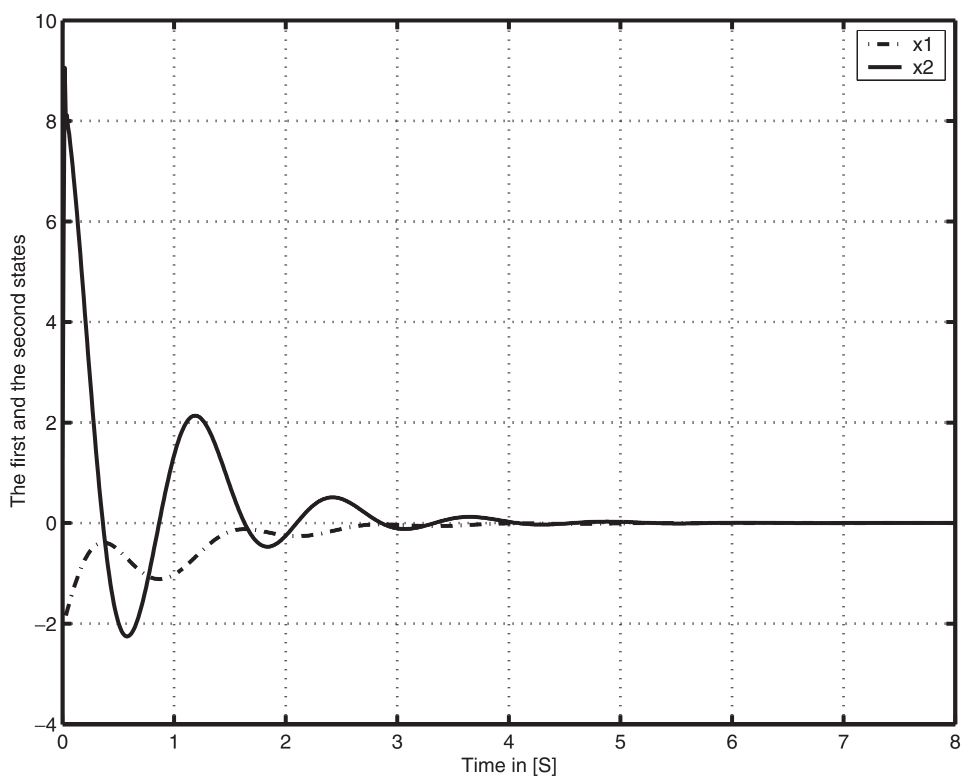

Figure 3. The states $x_1(k)$ and $x_2(k)$. Observer-based control.

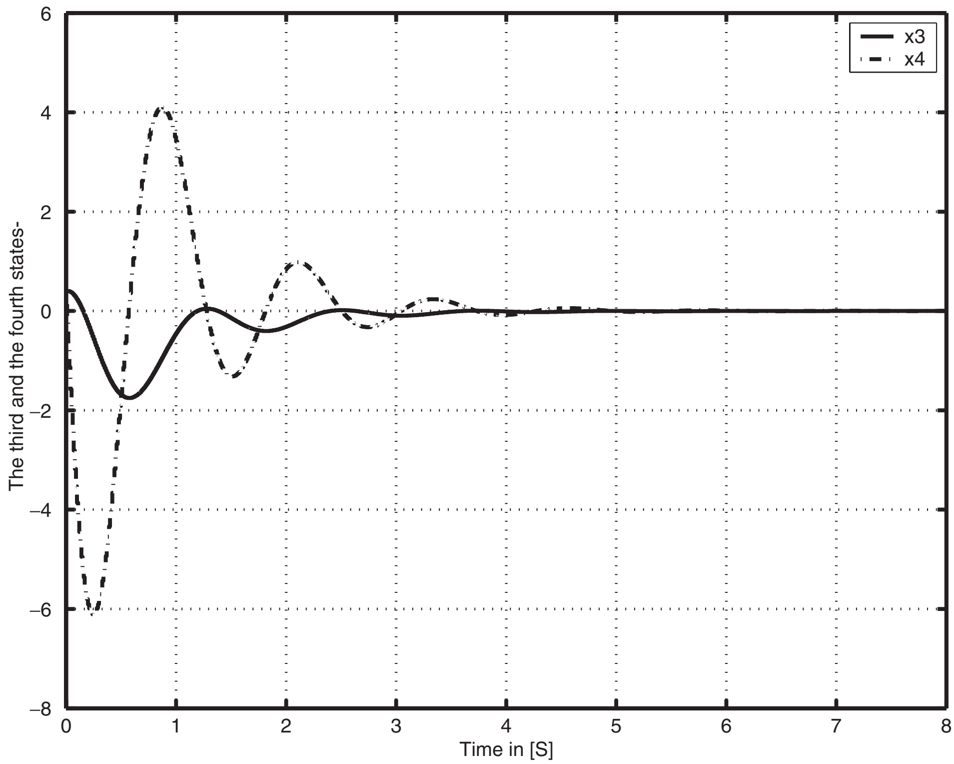

Figure 4. The states $x_3(k)$ and $x_4(k)$. Observer-based control.

to S_1 and S_2, μ_1, μ_2, and Y_2. This gives

$$S_1 = \begin{bmatrix} 1.1992 & 0.0369 & -0.3851 & 0.1501 \\ 0.0369 & 0.0091 & -0.0251 & 0.0039 \\ -0.3851 & -0.0251 & 0.5962 & -0.0074 \\ 0.1501 & 0.0039 & -0.0074 & 0.0533 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0.7983 & 0.0223 & -0.0495 & 0.0089 \\ 0.0223 & 0.8013 & -0.5368 & 0.0764 \\ -0.0495 & -0.5368 & 1.2189 & -0.2629 \\ 0.0089 & 0.0764 & -0.2629 & 0.1383 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} -0.7908 & -0.0073 \\ 0.3398 & -0.7284 \\ -0.1088 & 0.0895 \\ -0.0108 & -0.0030 \end{bmatrix}, \mu_1 = 0.6127,\ \mu_2 = 0.9342.$$

The performance of the observer-based controller is shown in figures 3–4.

5. Conclusion

A linear matrix inequality approach to design of observer-based controllers for Lipschitzian discrete-time non-linear systems is investigated. We showed that the proposed linear matrix inequalities are less restrictive than those proposed in the literature. The potential of the proposed techniques were demonstrated through an example of one-link flexible joint robot.

Acknowledgements

The authors would like to thank the associate editor and anonymous referees for their suggestions and valuable comments.

This work is supported by the National Natural Science Foundation of China under grant 50390063.

References

A.N. Atassi and H.K. Khalil, "Separation principle for the stabilization of a class of nonlinear systems", *IEEE Transactions on Automatic Control*, 44, pp. 1672–1687, 1999.

A. Balluchi, L. Benvenuti, M.D.D. Benedetto and A.L.S. Vincentelli, "A hybrid observer for the driveline dynamics", in *Proceedings of European Control Conference*, 2001, pp. 618–623.

A. Balluchi, L. Benvenuti, M.D.D. Benedetto and A.L.S. Vincentelli, "Design of observers for hybrid systems", in *Proccedings of Hybrid Systems: Computation and Control Conference*, California, March 2002, Vol. 2289, pp. 76–89.

S. Boyd, L.E. Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequality in Systems and Control Theory*, SIAM, Philadelphia, 1994.

G. Ciccarella, M.D. Mora and A. Germani, "Observers for discrete-time nonlinear systems", *Systems and Control Letters*, 20, pp. 373–382, 1993.

F.A. Cuzzola and M. Morari, "An LMI approach for H_∞ analysis and control of discrete-time piecewise affine systems", *International Journal of Control*, 75, pp. 1293–1301, 2002.

G. Ferrari-Trecate, D. Mignone and M. Morari, "Moving horizon estimation for hybrid systems", *IEEE Transactions on Automatic Control*, 47, pp. 1663–1676, 2002.

K. Gu, "An integral inequality in the stability problem of time-delay systems", in *Proceedings of the 39th IEEE Conference in Decision and Control*, Sydney, 2000, pp. 2805–2810.

A.M. Dabroom and H.K. Khalil, "Output feedback sampled-data control of nonlinear systems using high-gain observers", *IEEE Transactions on Automatic Control*, 46(11), pp. 1712–1725, 2001.

H.J.C. Huijberts, T. Lilge and H. Nijmeijer, "Nonlinear discrete-time synchronization via extended observers", *International Journal of Bifurcation and Chaos*, 11, pp. 1997–2006, 2001.

W. Lee and K. Nam, "Observer design for autonomous discrete-time nonlinear systems", *Systems and Control Letters*, 17, pp. 49–58, 1991.

Z.G. Li, C.Y. Wen and Y. Soh, "Observer-based stabilization of switching linear systems", *Automatica*, 39, pp. 517–524, 2003.

T.-L. Liao and N.-S. Huang, "An observer-based approach for chaotic synchronization with application to secure communications", *IEEE Transactions on Circuits and Systems-I: Fundamental theory and applications*, 46, pp. 1144–1150, 1999.

C.-H. Lien, "Robust observer-based control of systems with state perturbations via lmi approach", *IEEE Transactions on Automatic Control*, 49, pp. 1365–1370, 2004.

P.E. Moraal and J.W. Grizzle, "Observer design for nonlinear systems with discrete-time measurements", *IEEE Transaction on Automatic Control*, 40, pp. 395–404, 1995.

S. Pettersson and B. Lennartson, "Hybrid system stablity and robustness verification using linear matrix inequalities", *International Journal of Control*, 75, pp. 1335–1355, 2002.

A.Y. Pogromsky and H. Nijmeijer, "Observer-based robust synchronization of dynamical systems", *International Journal of Bifurcation and Chaos*, 8, pp. 2243–2254, 1998.

S. Raghavan and J.K. Hedrick, "Observer design for a class of nonlinear systems", *International Journal of Control*, 59, pp. 515–528, 1994.

K. Reif, S. Günther, E. Yaz and R. Unbehauen, "Stochastic stability of the discrete-time extended kalman filter", *IEEE Transactions on Automatic Control*, 44, pp. 741–728, 1999.

Y.K. Song and J. Grizzle, "The extended KALMAN filter as a local asymptotic observer for nonlinear discrete-time systems", *Journal of Mathematical Systems, Estimation and Control*, 5, pp. 59–78, 1995.