On numerical observers. Application to a simple academic adaptive control example

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Abstract. *Numerical differentiation techniques and the so-called* regulation model *method are used in the design of a compensation law which makes the system $\dot{y} = u + \theta\, y^2$ not only stable but follow exogenous inputs. The stabilization of this simple academic system, where θ is assumed to be a constant with unknown value, is known as challenging. The compensator is shown to comply with the following conditions: 1. the measurements are available only at sampling instants, 2. and the measurements are known through an additive random noise. The regulation model method consists in the specification of the way the plant output should follow the reference input while the numerical differentiation ingredients avail the observer design task. The main design parameters consist of those of a second order constant linear dynamics. There choices govern the stability of the compensated system, the noise filtering, and the speed at which the output is steered to the reference input. The stability analysis reveals the noteworthy facts that, for fixed design parameters, the compensator is able to handle a quite significant amount of measurements noise, as well as* slowly *varying θ.*

Keywords. *Observers; Nonlinear control; Numerical methods.*

The problem of stabilizing the simple academic system $\dot{y} = u + \theta\, y^2$, where θ is a *constant* parameter with *unknown* value, appears in the book [6] as an illustration of backstepping features.

The basic question on this system is to drive it to the steady state from arbitrary initial conditions under the mild assumption that the parameter θ is constant, the exact value of θ being unknown. The control law, $u = -\lambda\, y - \hat{\theta}\, y^2$, may come to mind in an implicit attempt to make the compensated system behave like $\dot{y} = -\lambda y$, asymptotically. The reader is referred to the above mentioned book for a thorough analysis based on backstepping techniques.

In the present paper, this compensation problem is examined in an alternative way which crucially uses numerical observers and a new design technique.

The compensation method is the following: choose a *reasonable* model (which will be called the *regulation model*) of what the compensated system is wanted to behave like, then derive (often, in a straightforward way) the compensation law which should be implemented in order to achieve the previous goal.

Since a behavior which is independent of the parameter θ is desired, the regulation model should be free of θ. This regulation model is, therefore, not really comparable to the equation, $\dot{y} = u + \theta\, y^2$, but to what will be called the *parameter free external behavior*, and is the equations of the system in which θ has been *eliminated*. The parameter free external behavior of the system happens to be of the second order, implying that any reasonable choice of the regulation model should be of the second order at least.

A linear regulation model of the second order is proposed, and shown to be tractable, leading to a linearizing compensation law.

Given that the parameter is *observable* with respect to u and y, the parameter free external behavior is *free* of the parameter, explicitly only. Implicitly, it incorporates the available information on the parameter, namely, the fact that the parameter is constant. When $y = 0$, the observability of θ becomes *singular*: no information on θ is available. How should the compensation law go through (or get around) this singularity?

Here comes into play one main point of the present work: instead of attempting an observation of θ the quantity $\bar{\theta} = \theta y$ is observed. The latter being observable with no singularity. It happens that the parameter free external behavior explicitly invokes $\bar{\theta}$ but not θ.

Such a compensation law is shown to crucially depend on an estimation of $\dot{y}$ from the samples of y; the information on the parameter θ is actually buried in the first order derivative of the output. Since classical observers (except the continuous differentiation one, as may be traced back to [5]) are likely to fail to provide such a derivative information, this simple example makes think that numerical observers [4] are worth consideration in the course of critical systems design. It may be argued that there probably is no "real" system governed by dynamics with such a hard nonlinearity. This is not known to the authors, the point being only the simplicity of the example (the computations are easy and short enough for the reader to follow and discuss them).

The numerical observer may be made quite sharp but, theoretically it will always generate a nonzero estimation error. Even if this error may reasonably be considered as negligible, when the measurements are noisy, it becomes necessary to perform some stability analysis of the compensated system with a numerical observer in the closed loop. This is done here, showing that the compensated system with the numerical observer presents a satisfactory practical stability property in the presence of substantial measurements noise.

1 The parameter free external behavior

Since the system

$$\dot{y} = u + \theta\, y^2 \tag{1}$$

is to be compensated in order to obtain a behavior which does not depend on the specific value of θ, provided that θ remains constant, the equations of the behavior of (1) for all values of θ need to be considered. They are called the *parameter free external behavior* of (1). Precisely, the parameter free external behavior is the result of the (differential) elimination of θ from equation (1), and, of course, the assumption $\dot{\theta} = 0$. The parameter free external behavior of (1) is easily found to be the *disjunction* of the following two sets of equations and inequations

$$\left\{ \begin{array}{l} y = 0, \\ u = 0, \end{array} \right. \qquad \left\{ \begin{array}{l} y\ddot{y} - \dot{u}y - 2(\dot{y} - u)\dot{y} = 0, \\ y \neq 0, \end{array} \right. \tag{2}$$

in other words, any triplet (u, θ, y) of real functions of the time (with θ constant) is a trajectory of system (1), if, and only if, the couple (u, y) satisfies at least one of the sets of equations and inequations in (2); see [2] for more details.

When $y = 0$ the differential equation $y\ddot{y} - 2\dot{y}(\dot{y} - u) - \dot{u}y = 0$ reduces to $0 = 0$, leaving u undefined. That is to say, the first set of two equations in (2) is there to specify what u should be in this case, according to the definition of the system.

Remark 1 *All the trajectories of* (1) *are also ones of*

$$\ddot{y} - \dot{u} - 2\overline{\theta}\dot{y} = 0, \tag{3}$$

where the differential rational function

$$\overline{\theta} = \theta\, y = \left\{ \begin{array}{ll} \dfrac{\dot{y} - u}{y} & \text{if} \quad y \neq 0, \\ 0 & \text{if} \quad y = 0, \end{array} \right. \tag{4}$$

is actually defined everywhere *(when $y = 0$ then $\overline{\theta} = 0 \times \theta = 0$). The converse may be seen to be true, i.e., any trajectory of* (3) *comes from one of* (1)*.*

Remark 2 *Now, if a compensation law is proven to make all the trajectories of* (3) *respect some given condition, $\mathcal{C}$, then, of course, the* same *compensation law, when applied to system* (1)*, will make all its trajectories respect the* same *condition $\mathcal{C}$.*

This remark allows to take (3) as the parameter free external behavior to be compensated in place of (2), thus, resolving the singularity of the observability of θ with respect to u and y, when $y = 0$. This pretended resolution of the observability singularity would have been complete if it were possible to implement f in the form of (4). Due to machine precision, and to the residual error of the numerical observer, such a rational function cannot be implemented, see §6 for details.

2 On the regulation model design technique

A compensation objective, stabilization say, on a given system may be laid down in at least two different ways. It may just be asked for the state to be steered to zero, leaving to the designer the choice of the way the state is actually driven to zero. But this same stabilization objective may also be specified along with the way the state should go to zero. This is done by providing the dynamics (differential equations) which must be observed while the state is going to zero. For instance, it may be asked to drive the system

$$\dot{y} = u$$

to rest according to a second order dynamics:

$$\ddot{y} + 2\xi\omega\dot{y} + \omega^2 y = 0.$$

Similarly, it might be asked to make the previous system follow an external signal v in such a way that the tracking error satisfies

$$(v - y)^{\bullet\bullet} + 2\xi\omega(v - y)^{\bullet} + \omega^2(v - y) = 0,$$

where $(\cdot)^{\bullet}$ and $(\cdot)^{\bullet\bullet}$ stand for the first and second time derivatives of the respective enclosed quantities. The previous two equations are instances of what is called a *regulation model*. The qualifier *regulation* does not confine to the classic regulator problem, but rather refers to the genesis of this idea.

In general, a regulation model is loosely defined to be any model of what might be the design objective.

The main point in setting design objectives in terms of regulation models is that *the* compensation law which achieves the goal is then often readily obtainable. For instance, to drive the latter integrator to rest according to the above mentioned second order dynamics, the compensator is readily given by $\dot{u} + 2\xi\omega u = -\omega^2 y$. And, the compensator making the same integrator follow the signal v is $\dot{u} + 2\xi\omega u = \ddot{v} + 2\xi\omega\dot{v} - \omega^2(v - y)$. A quite interesting feature of this is that a *nonlinear* regulation model may, as well, be specified for a linear plant if this is found to be more natural; the compensation law which achieves this condition is then most likely nonlinear.

Vaguely, if *proper* compensators are to be considered it does not make sense to require a regulation model of the first order when the plant is of the second order. That is to say, there is some *reasonable* way to choose a regulation model in order for the resulting compensation law to be able to be implemented.

For system (1) a regulation model free of the parameter is, therefore, of the second order, given the parameter free external behavior (3) of (1). But this is almost the only restriction on the choice of a regulation model.

3 A linear regulation model

Given the parameter free external behavior (3) one of the simplest regulation models which may be chosen is the following one

$$(v-y)^{\bullet\bullet} + 2\xi\omega(v-y)^{\bullet} + \omega^2(v-y) = 0, \qquad (5)$$

which linearly tracks a reference input v with design parameters ξ and ω. Comparing (3) and (5) readily yields

$$\dot{u} = -2\overline{\theta}\dot{y} + 2\xi\omega(v-y)^{\bullet} + \omega^2(v-y) + \ddot{v}. \qquad (6)$$

This compensator is *smooth* in principle, and in practice, it may be made behave as such by tuning the design parameters, adequately.

Remark 3 *When dealing with noisy measurements, as envisioned by the numerical observer theory below, it may be searched for a regularizing or smoothing parameter in order to obtain a sufficiently clean output and output derivative from the noisy measurements. It is noted here an alternative method to that design task: the previous compensation law is modified as following.*

$$\dot{u} = -2\lambda\overline{\theta}\dot{y} + 2\xi\omega(v-y)^{\bullet} + \omega^2(v-y) + \ddot{v}, \qquad (7)$$

where λ is a design parameter that will be called the regularizing parameter for the compensator. It's crucial role will be transparent from the stability analysis below.

4 On the numerical observer

The reader is referred to [4] for a preliminary discussion on numerical observers and to [1] for technical details about numerical spline approximation of an unknown function given its samples. For the sake of briefness, only details on how to obtain the approximating B-spline are provided below.

The main idea of numerical observers may be summarized as follows. The measurements vector, $y(t_k) = (y_1(t_k), y_2(t_k), \ldots, y_p(t_k))$, of a system being available at discrete instants, $\mathbf{t} = (t_k)_{k\in\mathbb{N}}, t_k < t_{k+1}$, the question of approximating the derivatives of each component of these measurements is answered to by providing an approximant $\hat{y}(t) = (\hat{y}_1(t), \hat{y}_2(t), \ldots, \hat{y}_p(t))$ and then just taking the derivatives of $\hat{y}(t)$ for those of $y(t)$. Then, if the system is observable in an appropriate sense, the state variable may be recovered from the measurements derivatives by means of nondifferential operations.

As earlier reported in [4] a whole bunch of approximation techniques may be used. A quite standard one in the numerical analysis literature is the polynomial spline (spline for short) approximation that has been preferred here for its availability.

Referring to de Boor's book, *the j-th B-spline of order ℓ for $\mathbf{t}$ is defined to be*

$$B_{j,\ell,\mathbf{t}}(t) = (t_{j+\ell} - t_j)\,[t_j, \cdots, t_{j+\ell}]\,(\cdot - t)_+^{\ell-1} \ (t \in \mathbb{R}),$$

where $(x)_+ = \max(x, 0)$ denotes the *truncation* function, and the symbol $[\tau_i, \tau_{i+1}, \cdots, \tau_{i+k}]\, g$ designates the k-th *divided difference of a function g at the points* $\tau_i, \tau_{i+1}, \cdots, \tau_{i+k}$. The j-th B-spline of order ℓ for $\mathbf{t}$ verifies

$$B_{j,\ell,\mathbf{t}}(t) = 0 \quad (t \notin [t_j, t_{j+\ell}]),$$

$$B_{j,\ell,\mathbf{t}}(t) > 0 \quad (t_j < t < t_{j+\ell}),$$

and

$$\sum_{j=k-\ell+1}^{l-1} B_{j,\ell,\mathbf{t}}(t) = 1 \quad (t_k < t < t_l).$$

A spline function of order ℓ for $\mathbf{t}$ is a linear combination

$$s = \sum_{j\in\mathbb{N}} \alpha_i B_{j,\ell,\mathbf{t}}$$

over $\mathbb{R}$ of B-splines of order ℓ for $\mathbf{t}$.

In a *real time* process a *finite* moving window, $\mathbf{t}_W = (t_{k-W}, t_{k-W-1}, \ldots, t_k)$, of measurement sampling instants is considered, where t_k stands for the current instant. In other words, $\mathbf{t}$ is finite, and, for the sake of simplicity of the notations, $\mathbf{t}_W$ will be shortened as $\mathbf{t}$, implicitly assuming the current instant to be W, and $B_{j,\ell,\mathbf{t}}$ will be designated by B_j. At the current instant, t_W, the derivatives must, therefore, be approximated by using the measurements samples contained in the window, $\mathbf{t}$. In the time interval $[t_0, t_W]$ only the B-splines $B_0, \ldots, B_W$ might be nonzero. The splines defined on the latter interval form a linear $\mathbb{R}$-vector space of dimension at most $W+1$. Again, in order to simplify the notations, one component, y_i, of the measurements vector will be considered and denoted by y.

One basic existence theorem in spline approximation states that *among all splines of order $\ell = 2\mathfrak{l}$ for the window $\mathbf{t}$, there is one, and only one, which minimizes the following performance index*

$$J = \lambda \sum_{k=0}^{W} \left(\frac{\overline{y}(t_k) - \hat{y}(t_k)}{\delta_k} \right)^2 + (1-\lambda) \int_{t_0}^{t_W} \hat{y}^{(\mathfrak{l})^2}(t) \mathrm{d}t$$

where $\overline{y}(t_k) = y(t_k) + \varepsilon(t_k)$ are the measurements of the time function $y(t)$ which are corrupted by a white noise $\varepsilon(t)$, and where λ, $0 \le \lambda \le 1$, and δ_k are the regularizing *parameters.* This spline will be called *smoothing.*

It realizes a compromise between the fidelity to the data and the smoothness of the spline. The order $\ell = 2\mathfrak{l}$ has to be chosen in such a way as it let the spline be sufficiently smooth.

Noting that

$$\sum_{k=0}^{W} \left(\frac{\overline{y}(t_k) - \hat{y}(t_k)}{\delta_k} \right)^2 = (\overline{y} - B\alpha)' D (\overline{y} - B\alpha)$$

where $B = [b_{i,j}]$, $b_{i,j} = B_i(t_j)$, and $\alpha = (\alpha_0, \ldots, \alpha_W)'$, and $\overline{y} = (\overline{y}_0, \ldots, \overline{y}_W)'$, and

$$\int_{t_0}^{t_W} \hat{y}^{(\mathfrak{l})^2}(t) dt = \alpha' R \alpha$$

where $R = [r_{i,j}]$,

$$r_{i,j} = \int_{t_0}^{t_W} B_i^{(\mathrm{I})}(t) B_j^{(\mathrm{I})}(t) \mathrm{d}t,$$

the above criterion may be rewritten in the form

$$J = \lambda\,(\overline{y} - B\alpha)' D\,(\overline{y} - B\alpha) + (1-\lambda)\,\alpha' R\,\alpha$$

where

$$D = \mathrm{diag}(\delta_1^2, \delta_2^2, \ldots, \delta_W^2)\,.$$

In other words, the problem of finding the coefficients α of the spline is one of least squares. The solution is readily

$$\alpha = (\lambda\, B\, D\, B' + (1-\lambda) R)^{-1}\, \lambda\, B\, D\, \overline{y}.$$

5 Stability analysis

Preliminary remarks on the stability of the system $\dot{y} = u + \theta y^2$ are in order. Assuming a nonzero initial state, from simple computations it may be noticed that the system escapes to infinity when there is an error on the estimation of θ, and when u is given a constant value. This is the case no matter how θ is estimated. Since the way any compensation law will be implemented is most like through a zero order holder (which maintains u constant during the sampling periods) the latter instability dictates a minimum duration between any two sampling instants. This, in part, explains the sampling period of 10^{-3} which will appear in the simulations below.

If v, $\dot{v}$, $\ddot{v}$, y and $\dot{y}$ were exactly known at each sampling instant, then the compensation law

$$\dot{u} = -2\lambda\,\overline{\theta}\dot{y} + 2\xi\omega(v-y)^{\bullet} + \omega^2(v-y) + \ddot{v} \qquad (7)$$

where λ is then naturally taken to be 1, would make the system

$$\dot{y} = u + \theta\, y^2 \qquad (1)$$

behave exactly as

$$(v-y)^{\bullet\bullet} + 2\xi\omega(v-y)^{\bullet} + \omega^2(v-y) = 0, \qquad (5)$$

(for all constant θ), and there would have been no need of stability analysis of the compensated system, the latter having been chosen asymptotically stable.

However, the numerical observer, no matter how accurate it is, will introduce an estimation error on $\dot{v}$, $\ddot{v}$, y, $\dot{y}$, and, consequently, on the input derivative $\dot{u}$. Therefore, a stability analysis for the compensated system is in order. As licit, v is supposed to be exactly known, but its derivatives have to be estimated along with y (in case of noisy measurements), and $\dot{y}$. In addition, as noted earlier the differential function f cannot be implemented in its form (4); the real life implementation of f will necessarily introduce an "error".

That is to say, instead of (7), system (1) is fed with the following

$$\begin{aligned}
\hat{\dot{u}} &= -2\lambda\left(\overline{\theta} + \varepsilon_{\overline{\theta}}\right)\left(\dot{y} + \varepsilon_{\dot{y}}\right) \\
&\quad +2\xi\omega\left(\dot{v} + \varepsilon_{\dot{v}} - \dot{y} - \varepsilon_{\dot{y}}\right) \\
&\quad +\omega^2\left(v - y - \varepsilon_y\right) + \ddot{v} + \varepsilon_{\ddot{v}}\,, \\
&= -2\lambda\overline{\theta}\dot{y} + \left(2\xi\omega + 2\lambda\varepsilon_{\overline{\theta}}\right)(v-y)^{\bullet} \\
&\quad + \left(\omega^2 + 2\lambda\theta\varepsilon_{\dot{y}}\right)(v-y) + \ddot{v} \\
&\quad -2\lambda\dot{v}\varepsilon_{\overline{\theta}} - 2\lambda\theta v\varepsilon_{\dot{y}} - 2\lambda\varepsilon_{\overline{\theta}}\varepsilon_{\dot{y}} \\
&\quad +2\xi\omega\left(\varepsilon_{\dot{v}} - \varepsilon_{\dot{y}}\right) - \omega^2\varepsilon_y + \varepsilon_{\ddot{v}}\,,
\end{aligned} \qquad (8)$$

where $t \mapsto \varepsilon_x(t)$ is the error made on the signal x in its real life implementation. The closed loop system equation then behaves according to

$$\ddot{y} \;=\; \hat{\dot{u}} + 2\theta y\dot{y},$$

that is,

$$\begin{aligned}
&(v-y)^{\bullet\bullet} + \left(2\xi\omega + 2\lambda\varepsilon_{\overline{\theta}}\right)(v-y)^{\bullet} \\
&+ \left(\omega^2 + 2\lambda\theta\varepsilon_{\dot{y}}\right)(v-y) = 2\lambda\varepsilon_{\overline{\theta}}\dot{v} + 2\lambda\theta\varepsilon_{\dot{y}}v \\
&-2\xi\omega\left(\varepsilon_{\dot{v}} - \varepsilon_{\dot{y}}\right) + 2\lambda\varepsilon_{\overline{\theta}}\varepsilon_{\dot{y}} + \omega^2\varepsilon_y - \varepsilon_{\ddot{v}}\,.
\end{aligned} \qquad (9)$$

The coefficients as well as the input of the regulation model (5) have thus been perturbed according to (9). The stability of the system closed with the numerical observer is then clarified by the following result.

Theorem 4 *If $t \mapsto \pi_1(t)$, $t \mapsto \pi_2(t)$, and $t \mapsto b(t)$ are bounded continuous functions of the time $t \in [0, \infty[$, with respective bounds, $|\pi_1(t)|, |\pi_2(t)| \leq k_1$ ($t \geq 0$), $|b(t)| \leq k$ ($t \geq 0$), then there exists a constant r (k_1, and k) such that, for values of ω,*

(i) *r is positive,*

(ii) *r is comparable to $k/(\xi\omega)$,*

(iii) *and the disk $||x|| \leq r$ of $\mathbb{R}^2$ is an attractive positively invariant set for the system*

$$\left\{\begin{array}{rcl}
\dot{x}_1 &=& x_2\,, \\
\dot{x}_2 &=& -\left(\omega^2 + \pi_1(t)\right) x_1 \\
&& -\left(2\xi\omega + \pi_2(t)\right) x_2 - b(t)\,.
\end{array}\right.$$

For a proof and further details, see [3].

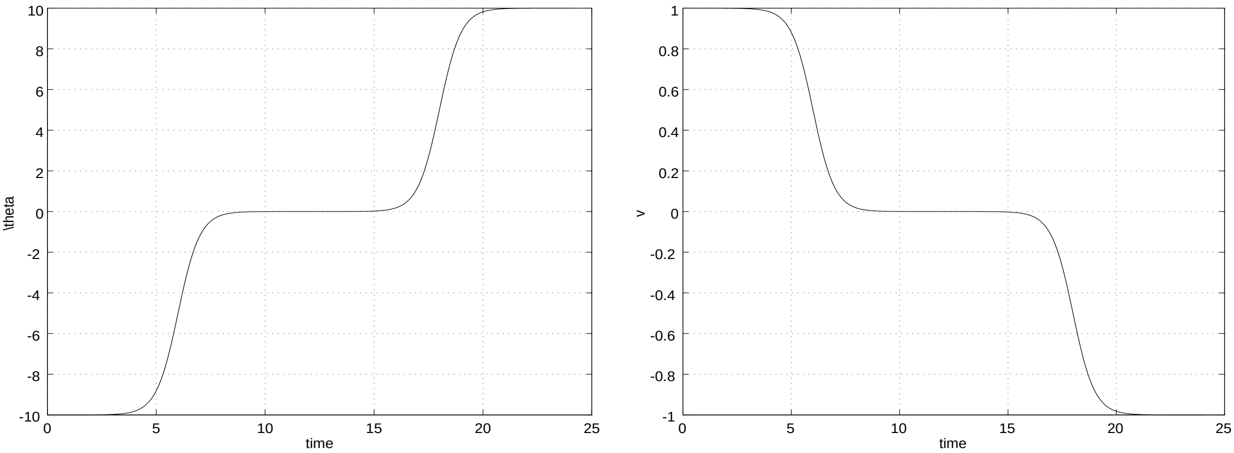
6 Simulations

In practice, due to the machine precision limitation, the differential rational function f in (4) will rather be implemented in the following way

$$\overline{\theta} = \left\{\begin{array}{ll}
\dfrac{\dot{y} - u}{y} & \text{if} \quad |y| > \varepsilon, \\[2ex]
\mathrm{sign}(\dot{y} - u) y & \text{if} \quad |y| \leq \varepsilon,
\end{array}\right.$$

where ε is a design parameter which should be set up according to the machine precision and the other design parameters.

In all the following simulations the true (unknown) plant parameter θ and the reference input are as follows.



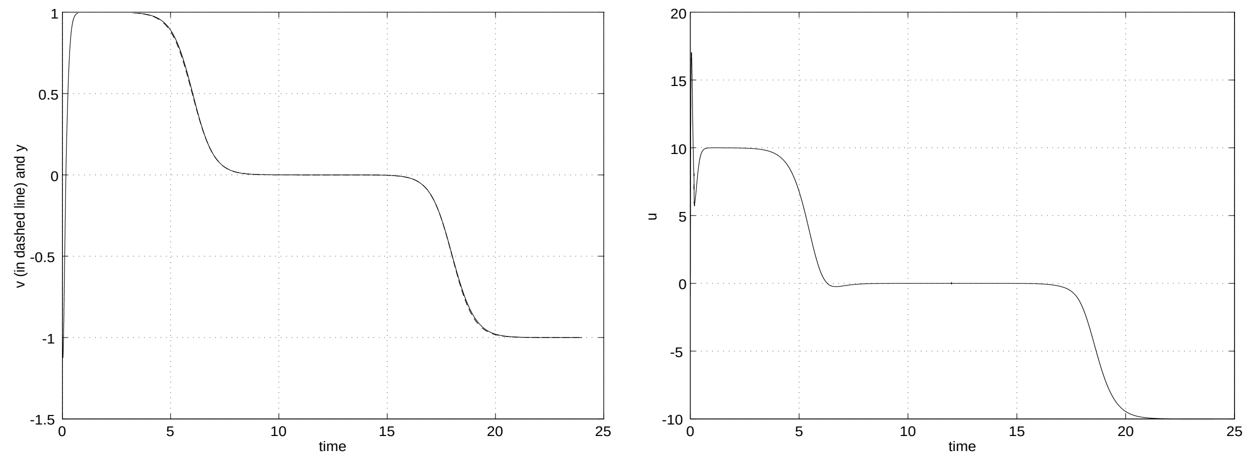
(a) θ variation

(b) v variation

Figure 1:

When the measurements are noise free, the regularizing parameter λ is taken to be 1. In order to assure closed loop stability ω has to be at least 4π. Then the other design parameters are as following: $\varepsilon = 10^{-3}$, $\xi = 1$. Measurements samples are equally spaced with a mesh size equal to .001 (It is not hard to see that for higher sampling periods with zero order holder for the control input the system goes unstable, no matter how the observer is designed). The compensator state has been initialized to 0. See Figure 2.

The overshooting at initial time is due to the delay for the observer to start functioning. It is noticeable that, even for the numerical observer which is the fastest observer that may be used (here, it needs less than 10 samples) the system has escaped so far the controller will spend time and energy control to compensate.

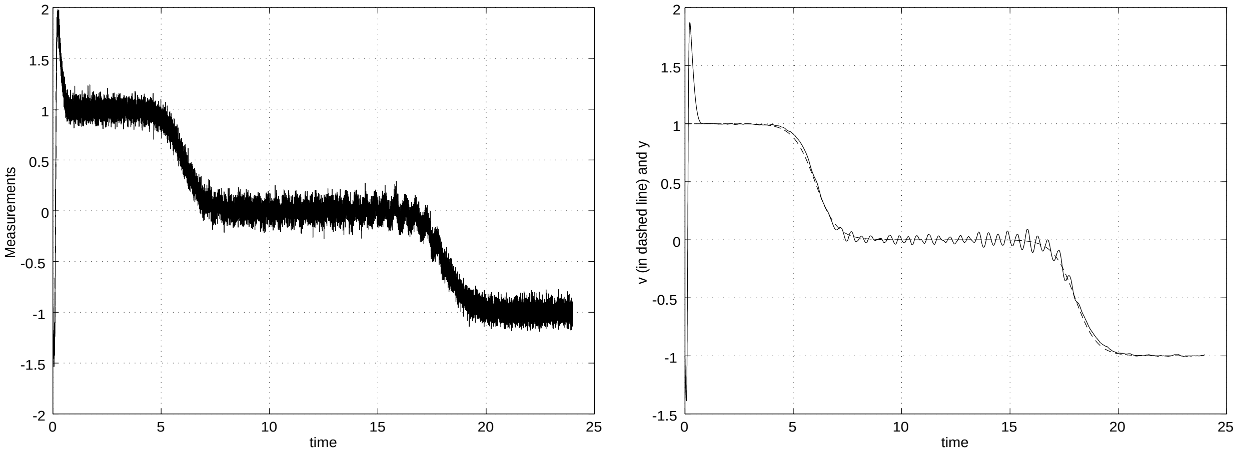


(a) y vs. v

(b) The control u

Figure 2:

Now a random (white) noise of, approximately, 20% magnitude is supposed to be added to y to stand for the noisy measurements.



(a) Measurements

(b) y vs. v

Figure 3:

It turns out that the overshoot reduction is in conflict against the filtering process. By requiring less filtering the overshooting may be eliminated.

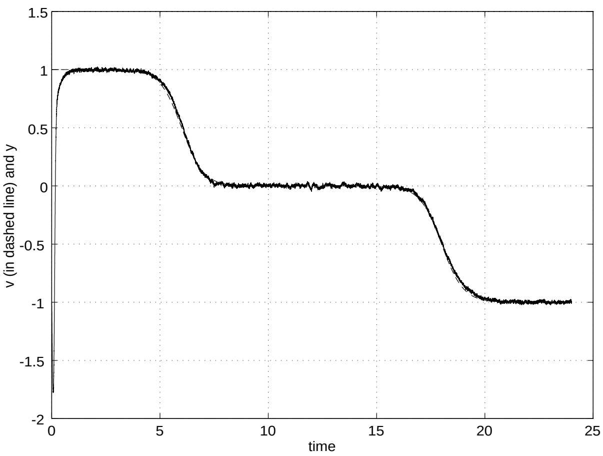


Figure 4: y vs. v

7 Concluding remarks

Constant values for ξ and ω have been used here to demonstrate the capability of the numerical observer and the compensator. It is apparent, however that, better performance may be obtained for the same compensator when the previous design parameters are made varying, according to the value of the signal to be followed, and, also, to the noise magnitude.

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