Robust Nonpeaking Algebraic Observers

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Abstract: - In this paper we propose a new robust scheme of nonpeaking nonlinear observers. The observation strategy is issued from differential algebra where the unmeasured states are given as outputs of a time-varying linear differentiator that guarantees robustness against measurement errors. We show that for a certain initial condition, the n-dimensional differentiator does not exhibit the peaking phenomenon, generally encountered in high-gain observer design. A discrete-time version is included to deal with sampled signals. The developed nonlinear observer reproduces the unmeasured states whatever the form of the nonlinear system that meets the algebraic observability conditions. *Key-Words:* Nonlinear observers; Differentiation; Lyapunov theory; Time-varying systems; Discrete-time systems.

1 Introduction

Nonlinear observer design has received widespread attention since the development of Kalman theory [1] and Luenberger observers [2]. However, the way to constructive nonlinear observer design still an open issue. In addition, robustness of high-gain observers with respect to measurement errors remains a difficult and a challenging task. Moreover, the peaking phenomenon, generally encountered in high-gain observer design is also an important problem that deters a lot of applications of the developed observer in closed-loop configurations.

In this paper we give a systematic procedure for designing optimal nonlinear observers subject to noisy measurements. Our approach is based upon differential algebra and enjoys the property of being easily implemented in either continuous-time or discrete-time manner. The nonlinear system is supposed to meet the algebraic observability conditions that translates the possibility of expressing the system states as a static functions that involve the input, the output, and finite number of their higher derivatives. Since the observation methodology turns on a fundamental problem of estimation of the higher derivatives of the system measured outputs, in this paper we plan to give a systematic procedure to conceive a stable high-order time-derivative tracker that decouple the effect of noise from the derivatives estimates. For a particular choice of initial conditions of the time-derivative tracker, we show that the higher derivatives are nonpeaking. Consequently, any state that involves these derivatives will be also nonpeaking. The main disadvantage of such observation methodology is the loss of the asymptotic convergence of the observer states. Under the assumption that the sates belong to an invariant predefined set, a positive parameter is selected to regulate the precision of the estimated states. We show that choosing this parameter sufficiently large does not affect the transient behavior of the observer sates.

The time-derivative estimation is achieved without any knowledge of the dynamical model of the signal to be differentiated. Ignoring the signal model while designing such differentiators is indispensable to conceive a nonpeaking system. In recent years, estimation of output derivatives has received a revival of interest in control and observation literatures, see for instance [3], [4], [5], [6], [7], [8], [9]. Both continuous-time and discrete-time high-gain observers have been applied to estimate the higher derivatives of given reference signals [9], [10], [8]. These estimates were used for several purposes as target tracking [10], semi-global stabilization of nonlinear systems [6] and nonlinear observer design [5]. Recall that high-gain observers are readily constructed as a copy of the dynamics of the original system with a proportional injection term that involves the system and the observer outputs. It is well-known that high-gain output injection is indispensable to defeat the inherent nonlinearities, however, this proportional injection arises two main drawbacks: the peaking phenomenon and noise amplification. Unfortunately, the high-gain observer is an observer which involves proportional injection term, and hence, the compromise between differentiation error, peaking, and noise filtering cannot be realized by the classical Luenberger observers.

In this paper we plan to reformulate the high-gain differentiation scheme by replacing the proportional P injection term with a multiple-integral time-varying injection term that involves the qth integral of the signal to be differentiated. Actually, the notion of adding an integral path is not quite new. The first idea of proportional integral PI observers has been proposed by Wojciechwski [11] and further developed by Beale and Shafai [12], and Niemann et al [13].The proposed time-derivative tracker differs from the conventional P and PI observers proposed in [12], [13], [14]. Our objective is to cancel the proportional term P from the observer dynamics and replace it by a novel injection term that depends upon the qth integral of the measured output. The static high-gain will be replaced by a time-varying one so as to avoid the peaking phenomenon in the sense of Sussmann and Kokotovic [15]. We show that the qI term permits to decouple the effect of noise from the derivative estimates and noise is filtered more and more by increasing the order of integration q. The discrete-time differentiation model is also included to deal with sampled signals. Throughout this paper, we note by $\mathbb{R}$ the set of real numbers, $\mathbb{Z}_{\geq 0}$ is the set of positive integer numbers, and $\delta_{i,j}$ stands for the kronecker symbol. $\|f(t)\|_\infty = \sup_{t\geq 0}|f(t)|$. $|f(t)|$ is the absolute value of the function $f(t)$. $y^{(i)}$ is the ith derivative of y. A' is the matrix transpose of A. $\|A\| = \max\left\{\sqrt{\lambda} : \lambda \text{ is the eigenvalue of } A^TA\right\}$. For any vector v, $\|v\|_S^2 = v'Sv$. $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{i,j}|$. $\lambda_{\min}(A)$: is the smallest eigenvalue of A. $\lambda_{\max}(A)$: is the largest eigenvalue of A. $\mathscr{S}^+(n, \mathbb{R})$ denotes the set of positive definite matrices of order n. ϵ is a small positive parameter and I is the identity matrix with appropriate dimension. $\mathrm{eig}(A)$ is the set of the eigenvalues of A. $\mathscr{M}(A)$ is the measure of the matrix A equal to $\lambda_{\max}\{(A + A')/2\}$. e^A is the matrix exponential.

2 Nonpeaking robust time-derivative tracker

In this section, we commence by developing the continuous-time nth order time-varying time-derivative tracker and then we deduce the discrete-time version of the observer by exact discretization. It will be highlighted that the peaking will be totaly removed in the first instants by a suitable choice of initial conditions, in the same time, the tracker remains robust against measurement errors that may occur at any moment.

2.1 Robustness against uncertainties

The objective is to conceive a time-varying nth order tracker that estimates the reliable higher-derivatives of a scalar output signal $y(t)$. We assume that there is no deterministic mathematical model for $y(t)$. Indeed, we can consider that the signal $y(t)$ is the output of the following system

$$\begin{array}{rcl} \dot{x}(t) & = & A\,x(t) + B\,y^{(n)}(t), \\ y(t) & = & Cx(t) + d(t), \end{array} \tag{1}$$

where $x = \left[\begin{array}{cccc} y & \dot{y} & \cdots & y^{(n-1)} \end{array}\right]'(t) \in \mathbb{R}^n$ is the state vector, and $y^{(n)}(t)$ is the unknown model input. The nominal matrices are defined as: $A \in \mathbb{R}^{n\times n}$: $(A)_{i,j} = \delta_{i,j-1}$, $1 \leq i,\, j \leq n$, $B \in \mathbb{R}^{n\times 1}$: $B_i = \delta_{n,i}$, $1 \leq i \leq n$ and $C \in \mathbb{R}^{1\times n}$: $C_i = \delta_{1,i}$, $1 \leq i \leq n$.

In references [5] and [9], we proposed a time-varying high-gain observer of the form

$$\begin{array}{c} \dot{\hat{\eta}}(t) = A\hat{\eta}(t) + P^{-1}(t)C'\left(y(t) - C\,\hat{\eta}(t)\right), \\ \dot{P}(t) = -\mu P(t) - A'P(t) - P(t)A + C'C, \\ P^{-1}(0) = \epsilon I,\; \mu \in \mathbb{R}_{>0}, \end{array} \tag{2}$$

to estimate the first $(n-1)th$ derivatives of $y(t)$. Recall that the static form of the last matrix differential equation $(-\mu P - A'P - PA + C'C = 0)$ has been proposed in [16] for nonlinear observer design where the system dynamics is assumed to be known. Although system (2) is a nonpeaking differentiation system, the presence of the proportional injection term $P^{-1}(t)C'\left(y(t) - C\,\hat{\eta}(t)\right) = P^{-1}(t)C'C\left(x(t) - \hat{\eta}(t)\right) + P^{-1}(t)C'd$ amplifies enormously the amount of noise for μ large. This means that whatever the method of calculation of the differentiation gain $P^{-1}(t)C'$, the tracker (2) could not decouple the effect of noise from the derivative estimates. For this reason, we reformulate the dynamics of the tracker as a time-varying observer of the form

$$\begin{array}{rcl} \dot{\xi}_1(t) & = & \xi_2(t) - k_{\xi_1}(t)\,\xi_1(t), \\ \dot{\xi}_2(t) & = & \xi_3(t) - k_{\xi_2}(t)\,\xi_1(t), \\ & \vdots & \\ \dot{\xi}_q(t) & = & y(t) - C\hat{x}(t) - k_{\xi_q}(t)\,\xi_1(t), \\ \dot{\hat{x}}(t) & = & A\,\hat{x}(t) - K_I(t)\,\xi_1(t), \end{array} \tag{3}$$

where $K_I(t) = \begin{bmatrix} k_1(t) & k_2(t) & \cdots & k_n(t) \end{bmatrix}' \in \mathbb{R}^n$ and $K_\xi = \begin{bmatrix} k_{\xi_1}(t) & k_{\xi_2}(t) & \cdots & k_{\xi_q}(t) \end{bmatrix}' \in \mathbb{R}^q$ are called herein the integral gain and the ξ-subsystem gain, respectively. The design of the time-varying observer gains is detailed in the following theorem.

Theorem 1 *Consider the time-varying linear system*

$$\begin{aligned} &\begin{bmatrix} \dot{\xi} \\ \dot{\hat{x}} \end{bmatrix}(t) = \left(\widetilde{A} - H^{-1}\widetilde{C}'\widetilde{C}\right)\begin{bmatrix} \xi \\ \hat{x} \end{bmatrix}(t) \\ &+ \begin{bmatrix} B_\xi \\ 0 \end{bmatrix} y(t), \\ &\dot{H}(t) = -\mu H(t) - \widetilde{A}'H(t) - H(t)\widetilde{A} + \widetilde{C}'\widetilde{C}, \end{aligned} \tag{4}$$

where μ is a sufficiently large positive constant, $A_\xi \in \mathbb{R}^{q\times q}$: $(A_\xi)_{i,j} = \delta_{i,j-1}$ is the anti-shift matrix, $\xi \in \mathbb{R}^q$ and $\hat{x} \in \mathbb{R}^n$ are the observer states, and the nominal matrices are defined as

$$\widetilde{A} = \begin{bmatrix} A_\xi & -B_\xi C \\ 0_{n\times q} & A \end{bmatrix} \in \mathbb{R}^{(n+q)\times(n+q)},$$

$$B_\xi = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{q\times 1},\ \widetilde{C} = \begin{bmatrix} C' \\ 0_{q\times 1} \end{bmatrix}' \in \mathbb{R}^{1\times n+q}. \tag{5}$$

Then for any uniformly bounded signal $y(t) \in \mathscr{C}^n$, measured with an error $d(t)$, such that $H(0) \gg I$ there exist a finite time T and two positive constants K_0, and K_1 such that

$$\sup_{t\geq T} \|\hat{x}(t) - x(t)\|_{H_\infty} \leq$$
$$\frac{K_0}{\mu} + K_1\left(\frac{\left|y^{(n)}(t)\right|}{\mu^{q+n+\frac{1}{2}}} + \frac{|d(t)|}{\mu^{q+\frac{1}{2}}}\right). \tag{6}$$

Proof. In the sequel the time variable t will be omitted for notation simplicity. From equation (4), we have $\begin{bmatrix} K_\xi(t) \\ K_I(t) \end{bmatrix} = H^{-1}(t)\widetilde{C}'$. The explicit solution of $H(t)$ is

$$H(t) = e^{-\mu t}e^{-\widetilde{A}'t}H(0)e^{-\widetilde{A}t}$$
$$+\int_0^t e^{-\mu(t-\tau)}e^{-\widetilde{A}'(t-\tau)}\widetilde{C}'\widetilde{C}e^{-\widetilde{A}(t-\tau)}\mathrm{d}\tau. \tag{7}$$

From the last equation, we see that $H(t)$ is always positive definite because (A, C) is an observable pair. After a finite time, H converges to a static matrix of the form

$$H_\infty = \int_0^\infty e^{-\mu(t-\tau)}e^{-\widetilde{A}'(t-\tau)}\widetilde{C}'\widetilde{C}e^{-\widetilde{A}(t-\tau)}\mathrm{d}\tau$$
$$= \frac{\left(\widetilde{H}_\infty\right)_{i,j}}{\mu^{i+j-1}},\ 1 \leq i,\ j \leq n+q, \tag{8}$$

where $\widetilde{H}_\infty$ is the solution of the matrix equation $-\widetilde{H}_\infty - \widetilde{A}'\widetilde{H}_\infty - \widetilde{H}_\infty\widetilde{A} + \widetilde{C}'\widetilde{C} = 0$. For notation simplicity H will stand for $H(t)$. Let $e = \hat{x} - x$ be the error between the state vectors of systems (1) and (4), and define

$$\widetilde{B}_\xi = \begin{bmatrix} B_\xi \\ 0_{n\times 1} \end{bmatrix},\ \widetilde{B} = \begin{bmatrix} 0_{q\times 1} \\ B \end{bmatrix},\ z = \begin{bmatrix} \xi \\ e \end{bmatrix}, \tag{9}$$

then

$$\dot{z} = \left(\widetilde{A} - H^{-1}\widetilde{C}'\widetilde{C}\right)z + \widetilde{B}_\xi\, d - \widetilde{B}y^{(n)}. \tag{10}$$

Let H_∞ be the solution of the following matrix equation

$$-\mu H_\infty - \widetilde{A}'H_\infty - H_\infty\widetilde{A} + \widetilde{C}'\widetilde{C} = 0, \tag{11}$$

and setting $V = z'H_\infty z$, $A_0 = \widetilde{A} - H_\infty^{-1}\widetilde{C}'\widetilde{C}$ and $\Delta H = H_\infty^{-1} - H^{-1}$, then

$$\begin{aligned} \dot{V} &= z'\left(A_0'H_\infty + H_\infty A_0 + 2H_\infty\Delta H\widetilde{C}'\widetilde{C}\right)z \\ &+ 2d\,\widetilde{B}_\xi'H_\infty z - 2y^{(n)}\,\widetilde{B}'H_\infty z. \end{aligned}$$

Using (11), we have

$$\begin{aligned} \dot{V} &\leq z'\left(-\mu H_\infty + 2H_\infty\Delta H\widetilde{C}'\widetilde{C}\right)z + 2d\widetilde{B}_\xi'H_\infty z \\ &- 2y^{(n)}\widetilde{B}'H_\infty z \\ &\leq -\left(\mu - 2\left\|H_\infty^{\frac{1}{2}}\Delta H\widetilde{C}'\widetilde{C}H_\infty^{-\frac{1}{2}}\right\|\right)V \\ &+ 2d\widetilde{B}_\xi'H_\infty z - 2y^{(n)}\widetilde{B}'H_\infty z \end{aligned} \tag{12}$$

Remark that H_∞ can be rewritten as $H_\infty = \mu D_\mu\widetilde{H}_\infty D_\mu$ such that

$$D_\mu = \mathrm{diag}\left[1/\mu,\ 1/\mu^2,\ \cdots,\ 1/\mu^{n+q}\right]. \tag{13}$$

Then we obtain the following bounds

$$\mu\,\underline{\lambda}\,\|D_\mu z\|^2 \leq V \leq \mu\,\bar{\lambda}\,\|D_\mu z\|^2, \tag{14}$$

such that $\underline{\lambda}$ and $\bar{\lambda}$ are the minimum and the maximum eigenvalues of $\widetilde{H}_\infty$, respectively. We have $\left\|\widetilde{B}_\xi'D_\mu\right\| = 1/\mu^q$, and $\left\|\widetilde{B}'D_\mu\right\| = 1/\mu^{q+n}$. This gives

$$\begin{aligned} 2d\,\widetilde{B}_\xi'H_\infty z &\leq 2\mu\bar{\lambda}\,|d|\left\|\widetilde{B}_\xi'D_\mu\right\|\|D_\mu z\| \\ &\leq \frac{C_1\,|d|}{\mu^{q-\frac{1}{2}}}\sqrt{V} \end{aligned} \tag{15}$$

where $C_1 = 2\bar{\lambda}/\sqrt{\underline{\lambda}}$ and

$$\begin{aligned} 2y^{(n)}\,\widetilde{B}'H_\infty z &\le 2\mu\bar{\lambda}\left|y^{(n)}\right|\left\|\widetilde{B}'D_\mu\right\|\|D_\mu z\| \\ &\le \frac{C_1\left|y^{(n)}\right|}{\mu^{q+n-\frac{1}{2}}}\sqrt{V} \end{aligned} \tag{16}$$

Inequality (12) becomes

$$\begin{aligned} \dot{V} &\le -\left(\mu - 2\left\|H_\infty^{\frac{1}{2}}\Delta H\widetilde{C}'\widetilde{C}H_\infty^{-\frac{1}{2}}\right\|\right)V \\ &+ \frac{C_1\left|y^{(n)}\right|}{\mu^{q+n-\frac{1}{2}}}\sqrt{V} + \frac{C_1\,|d|}{\mu^{q-\frac{1}{2}}}\sqrt{V}. \end{aligned} \tag{17}$$

From equations (11) and (4), the dynamics of the difference $G = H - H_\infty$ is $\dot{G} = -\mu G - \widetilde{A}'G - G\widetilde{A}$ which gives

$$G = e^{-\mu\,t}e^{-\widetilde{A}'\,t}G(0)e^{-\widetilde{A}\,t}. \tag{18}$$

We have $\left\|H_\infty^{\frac{1}{2}}\right\| \le \frac{\underline{c}}{\sqrt{\mu}}$ and $\left\|H_\infty^{-\frac{1}{2}}\right\| \le \bar{c}\mu^{n+q-\frac{1}{2}}$, such that $\bar{c} = \sqrt{n}\left\|\widetilde{H}_\infty^{-\frac{1}{2}}\right\|_\infty$, $\underline{c} = \sqrt{n}\left\|\widetilde{H}_\infty^{\frac{1}{2}}\right\|_\infty$, Moreover

$$\|G\| \le C_0 e^{-\mu\,t}\lambda_{\max}\left(e^{-\widetilde{A}'\,t}e^{-\widetilde{A}\,t}\right), \tag{19}$$

where $C_0 = \|G(0)\|$. Using

$$\begin{aligned} &\lambda_{\max}\left(e^{-\widetilde{A}'\,t}e^{-\widetilde{A}}\right) \\ &\quad \le e^{2\lambda_{\max}\left\{\left(\widetilde{A}+\widetilde{A}'\right)/2\right\}\,t} = e^{\sqrt{q+n}\,t}, \end{aligned}$$

then we obtain

$$\|G\| \le C_0 e^{-(\mu-\sqrt{q+n})\,t}. \tag{20}$$

Since $\left\|H^{-1}\right\| \le H_\infty^{-1}$ for all $H^{-1}(0) \le H_\infty^{-1}$, then

$$2\left\|H_\infty^{\frac{1}{2}}\Delta H\widetilde{C}'\widetilde{C}H_\infty^{-\frac{1}{2}}\right\| \le 2\left\|H_\infty^{-\frac{1}{2}}\,G\,H^{-1}H_\infty^{-\frac{1}{2}}\right\|$$

$$\le C_2\mu^{4(q+n)-2}e^{-(\mu-\sqrt{q+n})\,t},$$

where $C_2 = 2C_0\bar{c}^4$ does not depend on μ. Finally,

$$\begin{aligned} \dot{V} &\le -\left(\mu - C_2\mu^{4(q+n)-2}e^{-(\mu-\sqrt{q+n})\,t}\right)V \\ &+ \frac{C_1\left|y^{(n)}\right|}{\mu^{q+n-\frac{1}{2}}}\sqrt{V} + \frac{C_1\,|d|}{\mu^{q-\frac{1}{2}}}\sqrt{V}. \end{aligned} \tag{21}$$

Let $W = \sqrt{V}$, then for $t \ge T = \ln\left(2C_2\mu^{4(q+n)-2}\right)/(\mu - \sqrt{q+n})$, we have

$$\begin{aligned} \dot{W} &\le -\frac{\mu}{4}W \\ &+ \frac{C_1}{2}\left(\frac{\left|y^{(n)}\right|}{\mu^{q+n-\frac{1}{2}}} + \frac{|d|}{\mu^{q-\frac{1}{2}}}\right). \end{aligned} \tag{22}$$

Using the Gronwall-Bellman inequality, we get

$$\begin{aligned} W &\le e^{-\frac{\mu}{4}\,T}W(T) \\ &+ 2C_1\left(\frac{\left|y^{(n)}\right|}{\mu^{q+n+\frac{1}{2}}} + \frac{|d|}{\mu^{q+\frac{1}{2}}}\right). \end{aligned} \tag{23}$$

Put $e^{-\frac{\mu}{4}\,T}W(T) = \dfrac{K_0}{\mu}$, $K_1 = 2C_1$ and using the fact that $\|e\|_{H_\infty} \le \|z\|_{H_\infty}$, then we obtain

$$\sup_{t\ge T}\|\hat{x} - x\|_{H_\infty} \le \frac{K_0}{\mu} + K_1\left(\frac{\left|y^{(n)}\right|}{\mu^{q+n+\frac{1}{2}}} + \frac{|d|}{\mu^{q+\frac{1}{2}}}\right).$$

From the last inequality, we see that the effect of the perturbation d is attenuated more and more by increasing the order of integration q. In the next subsection, we prove that tracker (4) is a nonpeaking system if certain initial conditions are considered.

2.2 Peaking

We have seen from inequality (23) that noise reduction depends on the values of μ and q. In this subsection, we show that large values of μ do not affect the transient behavior of tracker (4). It means that choosing μ large augments the precision of tracker (4) and dwindles the effect of noise without defacing the transient behavior. We summarize the result in the following statement.

Theorem 2 *For μ large, system (4) is a nonpeaking differentiation observer for all $H(0) = \frac{1}{\epsilon_0}I \in \mathscr{S}^+(n+q,\,\mathrm{IR})$. ϵ_0 is a small positive parameter chosen in the interval $]0,\,1[$.*

Proof. From (4), we see that the tracker is a stable time-varying linear system perturbed by the input y. Let $\eta = \begin{bmatrix} \xi \\ \hat{x} \end{bmatrix}$ be the state vector of (4) for $y = 0$, then (4) is a nonpeaking system, in the sense of Sussmann and Kokotovic [15], if and only if the following system

$$\begin{cases} \dot{\eta} = \left(\widetilde{A} - H^{-1}\widetilde{C}'\widetilde{C}\right)\eta, \\ \dot{H} = -\mu H - \widetilde{A}'H - H\widetilde{A} + \widetilde{C}'\widetilde{C}, \end{cases} \tag{24}$$

is nonpeaking. Taking $V = \eta'H\eta$ as a Lyapunov function candidate to (24), then we get

$$\dot{V} \le -\mu V. \tag{25}$$

Then $V \leq e^{-\mu\, t} V(0)$, or

$$\|\eta\|^2 \leq \left(e^{-\mu\, t}\, \|H(0)\|\, \|\eta(0)\|^2\right) / \lambda_{\min}(H(t)). \quad (26)$$

Since

$$\lambda_{\min}(H(t)) \geq \lambda_{\min}\left(e^{-\mu t} e^{-\widetilde{A}'\, t} H(0) e^{-\widetilde{A}\, t}\right)$$
$$+\lambda_{\min}\left(\int_0^t e^{-\mu(t-\tau)} e^{-\widetilde{A}'(t-\tau)} \widetilde{C}' \widetilde{C} e^{-\widetilde{A}(t-\tau)} \mathrm{d}\tau\right),$$

and $\left(\widetilde{A},\, \widetilde{C}\right)$ is observable, then

$$\lambda_{\min}\left(\int_0^t e^{-\mu(t-\tau)} e^{-\widetilde{A}'(t-\tau)} \widetilde{C}' \widetilde{C} e^{-\widetilde{A}(t-\tau)} \mathrm{d}\tau\right) \geq \epsilon$$

$\forall t > 0$. Moreover

$$\lambda_{\min}\left(e^{-\mu t} e^{-\widetilde{A}' t} H(0) e^{-\widetilde{A} t}\right)$$
$$\geq \lambda_{\min}\left(e^{-\mu t} \cdot I\right) \lambda_{\min}\left(e^{-\widetilde{A}' t} H(0) e^{-\widetilde{A} t}\right).$$

Then

$$\lambda_{\min}(H(t)) \geq e^{-\mu t} \lambda_{\min}\left(e^{-\widetilde{A}' t} H(0) e^{-\widetilde{A} t}\right) + \epsilon \quad (27)$$

Using

$$\lambda_{\min}\left(e^{-\widetilde{A}' t} H(0) e^{-\widetilde{A} t}\right) \geq \frac{1}{\epsilon_0} \lambda_{\min}\left(e^{-\widetilde{A}' t} e^{-\widetilde{A} t}\right)$$
$$\geq \frac{1}{\epsilon_0} e^{-2\mathscr{M}(-\widetilde{A})\, t} = \frac{1}{\epsilon_0} e^{-\sqrt{n+q}\, t},$$

then

$$\|\eta\|^2 \leq \frac{e^{-\mu\, t}}{e^{-(\mu+\sqrt{n+q})\, t} + \epsilon\epsilon_0} \|\eta(0)\|^2. \quad (28)$$

For $t = -\ln((\mu\epsilon\epsilon_0)/\sqrt{n+q}))/(\mu + \sqrt{n+q})$, the function $\frac{e^{-\mu\, t}}{e^{-(\mu+\sqrt{n+q})\, t} + \epsilon\epsilon_0}$ reaches its maximum value

$$\max_{t \geq 0}\left(\frac{e^{-\mu\, t}}{e^{-(\mu+\sqrt{n+q})\, t} + \epsilon\epsilon_0}\right)$$
$$= \sqrt{n+q} \frac{\left(\frac{\mu\epsilon\epsilon_0}{\sqrt{n+q}}\right)^{\frac{\mu}{\mu+\sqrt{n+q}}}}{\epsilon\epsilon_0(\mu + \sqrt{n+q})}$$

For μ large, we have

$$\lim_{\mu\to\infty} \sqrt{n+q} \frac{\left(\frac{\mu\epsilon\epsilon_0}{\sqrt{n+q}}\right)^{\frac{\mu}{\mu+\sqrt{n+q}}}}{\epsilon\epsilon_0(\mu + \sqrt{n+q})}$$
$$= \lim_{\mu\to\infty} \frac{\mu}{\mu + \sqrt{n+q}} = 1$$

This implies that the peaking is absent for μ large. Finally, we conclude that the tracker does not exhibit any peaking in the first instants of the state reconstruction and behaves more resistant to any eventual perturbation that comes corrupting the reference signal y at any moment.

3 The discrete-time case

In most practical situations, signals are monitored in discrete-time manner. For this reason, it is recommended to conceive a discrete-time tracker that robustly estimate the higher derivatives of a given signal from its uncertain discrete-time samples. In this section, we show that by exact discretizing the continuous tracker (4), one could obtain a time-varying digital tracker that preserves all the advantages of the continuous-time tracker developed in section **??**. The breakdown of the digital tracker is given by the following theorem.

Theorem 3 *If the sampling period δ is chosen to satisfy the condition*

$$\max \mathrm{eig}\left(\sqrt{\sigma}\, e^{-\widetilde{A}\delta}\right) < 1, \quad (29)$$

then for all $H_0 \in \mathscr{S}^+(n+q, \mathbb{R})$, the state vector $\hat{x}_k$ of the discrete-time system

$$\begin{bmatrix} \xi_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} = \left(e^{\widetilde{A}\,\delta} - \delta\, H_k^{-1} \widetilde{C}' \widetilde{C} e^{\widetilde{A}\,\delta}\right) \begin{bmatrix} \xi_k \\ \hat{x}_k \end{bmatrix}$$
$$+\delta \begin{bmatrix} B_\xi \\ 0 \end{bmatrix} y_k,$$
$$H_{k+1} = \sigma e^{-\widetilde{A}'\,\delta} H_k e^{-\widetilde{A}\,\delta} + \delta\, \widetilde{C}' \widetilde{C},$$

robustly estimates the successive higher derivatives of the bounded signal $(y_k)_{k\in\mathbb{Z}_{\geq 0}}$ up to the order $n-1$. d_k is the measurement error and σ is called the smoothing parameter chosen in the $]0,\ 1[$. The nominal matrices are defined as in theorem 1.

Proof. For δ small enough such that we could neglect the terms of power δ^2, we have $e^{\widetilde{A}\delta} \sim I + \delta\, \widetilde{A}$, $\delta\, \widetilde{C} e^{\widetilde{A}\,\delta} \sim \delta\, \widetilde{C}$, $e^{-\widetilde{A}'\,\delta} \sim I - \delta \widetilde{A}'$, $e^{-A\delta} \sim I - \delta\, \widetilde{A}$. This gives

$$\begin{aligned} H_{k+1} &= \sigma\left(I - \delta \widetilde{A}'\right) H_k \left(I - \delta\, \widetilde{A}\right) + \delta\, \widetilde{C}' \widetilde{C} \\ &= \sigma H_k - \sigma\delta\, H_k \widetilde{A} - \sigma\delta \widetilde{A}' H_k \\ &+ \sigma\delta^2 \widetilde{A}' H_k \widetilde{A} + \delta\, \widetilde{C}' \widetilde{C} \end{aligned}$$

If we put $\sigma = 1 - \lambda$ such that $0 < \lambda < 1$ and neglecting the term of power δ^2, then for $\sigma \simeq 1$ and $\mu = \frac{\lambda}{\delta}$, we have

$$\lim_{\delta\to 0} \frac{H_{k+1} - H_k}{\delta} = \dot{H} = -\mu H - \widetilde{A}' H - H\widetilde{A} + \widetilde{C}' \widetilde{C}.$$

and

$$\lim_{\delta\to 0}\begin{bmatrix}\dfrac{\xi_{k+1}-\xi_k}{\delta}\\ \dfrac{\hat{x}_{k+1}-\hat{x}_k}{\delta}\end{bmatrix} = \left(\widetilde{A}-H^{-1}\widetilde{C}'\widetilde{C}\right)\begin{bmatrix}\xi\\ \hat{x}\end{bmatrix} + \begin{bmatrix}B_\xi\\ 0\end{bmatrix} y.$$

The condition of stability of the discrete-time tracker is a direct consequence of the stability of the discrete Lyapunov equation

$$H_{k+1} = \left(\sqrt{\sigma}e^{-\widetilde{A}'\delta}\right) H_k \left(\sqrt{\sigma}e^{-\widetilde{A}\delta}\right) + \delta\,\widetilde{C}'\widetilde{C}. \quad (30)$$

Then the matrix H_k could be written as

$$H_k = \delta\sum_{k=0}^{k}\sigma^k\left[e^{-\widetilde{A}'\delta}\right]^k\widetilde{C}'\widetilde{C}\left[e^{-\widetilde{A}\delta}\right]^k. \quad (31)$$

Since $\widetilde{A}$ is nilpotent for $k \geq n+q$, then

$$e^{-\widetilde{A}'\delta} = \sum_{i=0}^{\infty}(-1)^i\frac{\left(\widetilde{A}'\delta\right)^i}{i!} = \sum_{i=0}^{n+q-1}(-1)^i\frac{\left(\widetilde{A}'\delta\right)^i}{i!}; \qquad \text{with } \widetilde{A}^0 = I.$$

This gives

$$\delta\left[e^{-\widetilde{A}'\delta}\right]^k\widetilde{C}'\widetilde{C} = \delta\sum_{i=0}^{n+q-1}(-1)^i\frac{\left(k\,\widetilde{A}'\delta\right)^i}{i!}\widetilde{C}'\widetilde{C},$$

then

$$\delta\left[e^{-\widetilde{A}'\delta}\right]^k\widetilde{C}'\widetilde{C}\left[e^{-\widetilde{A}\delta}\right]^k = \delta\sum_{i=0}^{n+q-1}\sum_{j=0}^{n+q-1}\frac{(-k)^{i+j}}{i!j!}\left(\widetilde{A}'\delta\right)^i\widetilde{C}'\widetilde{C}\left(\widetilde{A}\delta\right)^j$$

Consequently, using (31), we have the expression of H_∞

$$\delta\sum_{k=0}^{\infty}\sigma^k\sum_{i=0}^{n+q-1}\sum_{j=0}^{n+q-1}\frac{(-k)^{i+j}}{i!j!}\left(\widetilde{A}'\delta\right)^i\widetilde{C}'\widetilde{C}\left(\widetilde{A}\delta\right)^j \quad (32)$$

Since $\sigma < 1$, then the infinity sum converges to a constant matrix that depends on σ and δ. The user can use the properties of the sums of geometric series to determine the final expression of H_∞.

In order to highlight the correspondence between the developed discrete-time tracker and the classical IIR differentiators, we shall omit the ξ-subsystem from the structure of the tracker (30), then its dynamics reduces to the following system:

$$\begin{cases}\hat{x}_{k+1} = e^{A\delta}\hat{x}_k + \delta H_k^{-1}C'\left(y_k - Ce^{A\delta}\hat{x}_k\right),\\ H_{k+1} = \sigma e^{-A'\delta}H_k e^{-A\delta} + \delta C'C.\end{cases} \quad (33)$$

H_k in the last system is $n\times n$ matrix. By taking the special case $n = 3$, we obtain

$$H_\infty = \sum_{k=0}^{\infty}\sigma^k\sum_{i=0}^{2}\sum_{j=0}^{2}\frac{(-k)^{i+j}}{i!j!}(A'\delta)^i\delta C'C(A\delta)^j = \sum_{k=0}^{\infty}\sigma^k\begin{pmatrix}\delta & -k\,\delta^2 & \frac{k^2\delta^3}{2}\\ -k\,\delta^2 & k^2\delta^3 & -\frac{k^3\delta^4}{2}\\ \frac{k^2\delta^3}{2} & -\frac{k^3\delta^4}{2} & \frac{k^4\delta^5}{4}\end{pmatrix}$$

which is equal to

$$\begin{pmatrix}\frac{\delta}{1-\sigma} & -\frac{\delta^2\sigma}{(1-\sigma)^2} & \frac{1}{2}\frac{\delta^3\sigma(\sigma+1)}{(1-\sigma)^3}\\ -\frac{\delta^2\sigma}{(1-\sigma)^2} & \frac{\delta^3\sigma(\sigma+1)}{(1-\sigma)^3} & -\frac{1}{2}\frac{\delta^4\sigma(\sigma^2+4\sigma+1)}{(1-\sigma)^4}\\ \frac{1}{2}\frac{\delta^3\sigma(\sigma+1)}{(1-\sigma)^3} & -\frac{1}{2}\frac{\delta^4\sigma(\sigma^2+4\sigma+1)}{(1-\sigma)^4} & \frac{1}{4}\frac{\delta^5\sigma(\sigma^3+11\sigma^2+11\sigma+1)}{(1-\sigma)^5}\end{pmatrix}$$

Then

$$\delta H_\infty^{-1}C' = \begin{pmatrix}(1-\sigma)(\sigma^2+\sigma+1)\\ \frac{3}{2}(\sigma-1)^2(\sigma+1)/\delta\\ (1-\sigma)^3/\delta^2\end{pmatrix}. \quad (34)$$

The resulting $z-$transfer functions of the tracker (33) (for $n = 3$) are:

$$\frac{\widehat{X}_1(z)}{Y(z)} = \frac{\left(1-\sigma^3\right)z^2 + \left(3\,\sigma^3 - 3\,\sigma\right)z - 3\,\sigma^3 + 3\,\sigma^2}{z^3 + (3\,\sigma-6)\,z^2 + (6-3\,\sigma^2)\,z - 2 + \sigma^3} \quad (35)$$

$$\frac{\widehat{X}_2(z)}{Y(z)} = \frac{(z-1)}{2\delta}\times \frac{\left(-3\,\sigma + 3\,\sigma^3 + 3 - 3\,\sigma^2\right)z - 1 - 5\,\sigma^3 + 9\,\sigma^2 - 3\,\sigma}{z^3 + (3\,\sigma-6)\,z^2 + (6-3\,\sigma^2)\,z - 2 + \sigma^3} \quad (36)$$

$$\frac{\widehat{X}_3(z)}{Y(z)} = \frac{(z-1)^2}{\delta^2}\frac{1-\sigma^3+3\,\sigma^2-3\,\sigma}{z^3 + (3\,\sigma-6)\,z^2 + (6-3\,\sigma^2)\,z - 2 + \sigma^3} \quad (37)$$

From (36), and (37) we see the forward difference formulas $\frac{(z-1)}{\delta}$ and $\frac{(z-1)^2}{\delta^2}$, issued from classical numerical differentiation, followed by an IIR discrete filters.

4 Observer design

First, let us begin by giving some important definitions.

Definition 1 *Consider the nonlinear system described by the following dynamic equations*

$$\begin{cases} \dot{x}(t) = f(x(t),\, u(t)), \\ y(t) = h(x(t)), \end{cases} \tag{38}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ *is continuously differentiable and satisfies* $f(0,\,0) = 0$. $x(t) \in \mathbb{R}^n$ *is the state vector,* $u(t) \in \mathbb{R}^m$ *is the input vector, and* $y(t) \in \mathbb{R}$ *is a smooth nonsingular output. We assume that* $y(t)$ *and* $u(t)$ *are continuously differentiable for all* $t \geq 0$. *System (38) is said to be algebraically observable if there exist two positive integers* μ *and* ν *such that*

$$x(t) = \phi\left(y,\, \dot{y},\, \ddot{y}, \cdots,\, y^{(\mu)},\, u,\, \dot{u},\, \ddot{u}, \cdots,\, u^{(\nu)}\right)(t), \tag{39}$$

where $\phi(\cdot) : \mathbb{R}^{\mu+1} \times \mathbb{R}^{(\nu+1)m} \mapsto \mathbb{R}^n$ *is a differentiable vector valued nonlinearity of the inputs, the outputs, and their derivatives.*

Notice that the last definition has been introduced in reference [17] to characterize the *uniform complete observability*. Recall that for nonlinear systems, there exists a set of control inputs which renders system (38) unobservable. We refer the reader to [18] for introductory discussions of this problem. For our case, we define this class of bad inputs as follows.

Definition 2 *System (38) is algebraically observable for any input, if the vector valued*

$$x(t) = \phi\left(y,\, \dot{y},\, \ddot{y}, \cdots,\, y^{(\mu)},\, u,\, \dot{u},\, \ddot{u}, \cdots,\, u^{(\nu)}\right)(t),$$

is defined on $\mathbb{R}^{\mu+1} \times \mathbb{R}^{(\nu+1)m} \mapsto \mathbb{R}^n$ *for all* $u \in \mathscr{U}$. *We call* $\mathscr{U}$ *the set of continuously differentiable control inputs for which the state vector (39) is defined everywhere, and we note* $\mathscr{U}^\star$, *the set of bad inputs that makes (39) singular.*

In this section we show how to use the differentiation observer as a nonlinear observer. For this purpose, consider the nonlinear system known by the name of the duffing oscillator

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - x_1^3, \\ y &= x_1 + x_2, \end{cases} \tag{40}$$

where $x = x(t)$ is the state vector and $y = y(t)$ is a scalar output. System (40) is algebraically observable, i.e.,

$$\begin{cases} x_1 &= \dfrac{\left(-3\dot{y} + \ddot{y} + 4y + y^3\right) y}{2y^3 - \ddot{y} + 5y}, \\ x_2 &= \dfrac{\left(3\dot{y} - 2\ddot{y} + y + y^3\right) y}{2y^3 - \ddot{y} + 5y}, \end{cases} \tag{41}$$

According to the above definition (see Eq. (41)), the nonlinear system is observable. The trajectory of the system states are uniformly bounded. One can take the following Lyapunov function candidate

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4, \tag{42}$$

and show that $\dot{V} = 0$, i.e., $V = C$ is a constant Lyapunov function. This implies that the first derivatives of y are also bounded. For μ sufficiently large, the robust q-integral nonpeaking observer is readily constructed as

$$\begin{aligned} \hat{x}_1 &= \frac{\left(-3\dot{y} + \hat{\eta}_3 + 4y + y^3\right) y}{2y^3 - \hat{\eta}_3 + 5y}, \\ \hat{x}_2 &= \frac{\left(3\hat{\eta}_2 - 2\hat{\eta}_3 + y + y^3\right) y}{2y^3 - \hat{\eta}_3 + 5y}, \\ \begin{bmatrix} \dot{\xi} \\ \dot{\hat{\eta}} \end{bmatrix} &= \left(\widetilde{A} - H^{-1}\widetilde{C}'\widetilde{C}\right) \begin{bmatrix} \xi \\ \hat{\eta} \end{bmatrix} + \begin{bmatrix} B_\xi \\ 0 \end{bmatrix} y, \\ \dot{H} &= -\mu H - \widetilde{A}'H - H\widetilde{A} + \widetilde{C}'\widetilde{C},\ H^{-1}(0) = \epsilon I, \end{aligned} \tag{43}$$

where the differentiation ξ- and $\hat{\eta}$-subsystems are defined as in theorem 1 and the dimension of the $\hat{\eta}$ is greater or equal to 3.

Remark 1 *The design of the robust algebraic observer is not limited to bounded state nonlinear systems, the reader is referred to the reference [19] to see how to encounter this problem by change of coordinate.*

5 Conclusion

In this paper we introduced a novel form of robust nonpeaking observers for nonlinear systems that verify the algebraic observability conditions. The novelty of the proposed observers consists in replacing the proportional output error by a q-integral time-varying injection term in the differentiator dynamical equations. For a particular choice of initial conditions, we showed that the continuous-time tracker is an arbitrary-order differentiation system that does not exhibit the peaking phenomenon. By increasing the order q of the integral path, noise is more attenuated and the observer remains robust against any perturbation that may attach to the signal to be differentiated. A discrete-time version of the differentiation scheme is included to deal with digital signals. The nice properties of the proposed observers favorite their applications in others numerous research areas as target tracking and semi-global stabilization of nonlinear systems.

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