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Automatica 43 (2007) 1432–1441
www.elsevier.com/locate/automatica

Brief paper

Circle-criterion approach to discrete-time nonlinear observer design☆

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Received 1 May 2006; received in revised form 18 January 2007; accepted 21 January 2007
Available online 12 June 2007

Abstract

This paper addresses the design of discrete-time nonlinear observers through the circle criterion. The new design method is mainly devoted to either globally Lipschitz systems or bounded-state systems whose nonlinearities can be decomposed into a linear combination of positive-slope nonlinearities. The observer design is not restricted to systems with positive-slope nonlinearities, but it encompasses systems with non-positive-slope nonlinearities too. Stability conditions of the observation error are given in terms of numerically tractable linear matrix inequalities. Illustrative examples are presented in order to highlight the main features and advantages of the new proposed technique over some classical designs.

Keywords: Discrete-time nonlinear observers; Circle criterion; Linear matrix inequalities (LMIs); Sampled systems; Multiple-output-injection observers

1. Introduction

In contrast to linear dynamical systems, there is no generic procedure to design a state observer for a given nonlinear model. However, solutions do exist for various cases. In the last decades, nonlinear observer design has been thoroughly investigated and quite successfully design methods are scattered in the vast literature related to this area. Available techniques for the design of nonlinear observers are broadly classified into different groups. First, high-gain observers based on pole-placement algorithms as in Thau (1973), Rajamani (1998), Raghavan and Hedrick (1994) and Tornambè (1992), Lyapunov-based design methods as in Arcak and Kokotović (2001), Fan and Arcak (2003) and Kazantzis and Kravaris (1998), geometric algorithms as in Glumineau, Moog, and Plestan (1996), Krener and Respondek (1985), Xia and Gao (1989) and Bestle and Zeitz (1983), sliding-modes design procedures as in Yaz and Azemi (1993) and Slotine, Hedrick, and Misawa (1987), algebraic techniques as in Ibrir (2003) and numerical procedures as in Moraal and Grizzle (1995).

The idea of transforming a nonlinear system into observable canonical forms has been widely used as a key solution to solve nonlinear observation issues, see e.g., Califano, Monaco, and Normand-Cyrot (2003). However, the existence of such state transformations that bring the system to some canonical forms of observation is generally attached to complex conditions that cannot always be verified by existing physical systems. In case where the system fails to be put in certain canonical forms, the construction of a high-gain observer turns out to be useful, see e.g., Lee and Nam (1991), Ciccarella, Mora, and Germani (1993), Raghavan and Hedrick (1994), Rajamani (1998), Reif, Günther, Yaz, and Unbehauen (1999) and Ibrir, Xie, and Su (2005). However, this standard approach which uses a copy of the system dynamics with a unique output correction term may fail due to the limitation of the linear-output-injection term which is basically conceived to defeat the adverse nonlinearities. In our opinion, the conservatism of high-gain observers is mainly due to the fact that nonlinearities are viewed as a system uncertainty and their structures are not exploited to reduce the complexity of the observation problem. The reader is referred to Xie and Guo (2000) for more details on the limitations of feedback in the presence of uncertainties and how can the capability of feedback be enhanced if a priori information about the system structure is available. For further details on how to characterize the relation between the distance to unobservability and the Lipschitz constants of nonlinearities,

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Tongwen Chen under the direction of Editor Ian Petersen.

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doi:10.1016/j.automatica.2007.01.012

the reader can also see Raghavan and Hedrick (1994), Rajamani and Cho (1998) and Aboky, Sallet, and Vivalda (2002) and the references therein. Besides all these difficulties, discrete-time implementation of high-gain observers is generally raised as a difficult issue since the stability of the observation error cannot be preserved under arbitrary sampling, see e.g., Dabroom and Khalil (1999), Arcak and Nešić (2004) and Ren and Guo (2005). In Ren and Guo (2005) the authors established an impossibility theorem that states that the class of uncertain nonlinear systems cannot be stabilized globally by any sampled-data feedback law whenever the sampling rate exceeds the value $4.757/\gamma$ where γ is the slope of the uncertain function. As a result, these particular problems call for a wide range of new theories, methodologies and techniques to enable synthesis and stability improvement of sampled-data systems. In this paper, we exploit the circle criterion in discrete time to give an extension of the works given in Arcak and Kokotović (2001) and Fan and Arcak (2003) to multi-variables discrete-time nonlinear systems. In this paper, we focus on the design of discrete-time nonlinear observers in an attempt to answer the following question: "given a discrete-time nonlinear system with either positive- and non-positive-slope nonlinearities, how to exploit the structure of nonlinearities in order to set up a converging observer with less conservative conditions". To answer this question, the developed design method in discrete time slightly differs from that developed in the continuous-time case (Fan & Arcak, 2003), in the sense that either positive- and non-positive-slope nonlinearities are tolerated. First, we begin by analyzing the discrete-time circle-criterion observer for globally Lipschitz systems. In an attempt to overcome the limitation of discrete-time high-gain observers that use a unique output injection term to feedback the observer, the system nonlinearity is decomposed into a linear combination of positive-slope nonlinearities. Subsequently, a nonlinear observer is conceived with nonlinear multiple-output-injection terms so as to make the observation error globally stable for any initial condition.

The second part of this paper is devoted to the observation of bounded-state systems whose nonlinearities have not a priori bounded slopes. Motivated by the results given in Shim, Son, and Seo (2000), we derive linear matrix inequality (LMI)-based conditions that ensure the existence of a semi-globally convergent observer for the bounded-state system. The main difference between our discrete-time design method and that proposed in continuous time (Shim et al., 2000) is that the system being considered is not in certain canonical forms and nonlinearities are saturated by a *new smooth saturation function* that preserves the differentiability of the saturated functions. We stress that the semi-global stability of the observer error does not restrict the initial conditions of the observer, but only the initial conditions of the system. Finally, the problem of estimating and enlarging the domain of observation is investigated in LMI framework. Throughout this paper, we note by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbf{0}$ and I the set of natural numbers, the set of integer numbers, the set of real numbers, the null matrix and the identity matrix of appropriate dimensions, respectively. The notation $A > 0$ (resp. $A < 0$) means that the matrix A is positive definite (resp. negative definite). A' is the matrix transpose of A. "$\star$" is used to notify an element which is induced by transposition. $\triangleq$ stands for an equality by definition. $\circ$ stands for the composition operator of functions. $f^{(-1)}(x)$ is the inverse function of the scalar function $f(x)$. $|\cdot|$ stands for the absolute value.

2. Circle-criterion-observer design in discrete time

In the last decades several works have been devoted to multivariable generalization of the discrete-time circle criterion, see e.g., Richter and Misawa (2003), Haddad and Bernstein (1994) and Wu (1967). Other techniques as the Tsypkin criteria have been also employed to derive stability conditions of the feedback interconnection of linear discrete-time systems and multivariable memoryless nonlinearities (Kapila & Haddad, 1996). In this section, we deal with the dual problem that consists of designing discrete-time nonlinear observers through the circle criterion. Our primary goal is to give an extension of the result given in Fan and Arcak (2003) to discrete-time nonlinear systems of the form

$$x_{k+1} = Ax_k + \sum_{i=1}^{\mu} G_i f_i(H_i x_k) + \psi(u_k, y_k),$$
$$y_k = Cx_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the system input and $y_k \in \mathbb{R}^p$ is the system output. The nominal matrices $A \in \mathbb{R}^{n\times n}$, $(G_i)_{1\leqslant i\leqslant \mu} \in \mathbb{R}^{n\times 1}$, $(H_i)_{1\leqslant i\leqslant \mu} \in \mathbb{R}^{1\times n}$ and $C \in \mathbb{R}^{p\times n}$ are constant known matrices. We assume that the pair (A, C) is observable. The term $\psi(u_k, y_k)$ is an arbitrary real-valued vector that depends on the system inputs and outputs and $(f_i(H_i x_k))_{1\leqslant i\leqslant \mu}$ are the system nonlinearities verifying the following growth conditions:

$$\frac{\mathrm{d} f_i(s)}{\mathrm{d}s} \geqslant 0, \quad 1\leqslant i\leqslant \mu, \quad \forall s \in \mathbb{R}. \tag{2}$$

Systems of form (1) may represent the dynamics of pure[1] discrete-time systems or the Euler discrete-time approximation of continuous-time systems studied in Fan and Arcak (2003). However, the new representation (1) may include more general nonlinearities which may not have positive slopes, see Example 2 for more details. We assume that the slope of nonlinearities does not exhibit an escape to infinity in finite time. Therefore, two different classes of systems are studied: globally Lipschitz systems and bounded-state nonlinear systems that have not a priori bounded slopes. If the slopes of nonlinearities escape to infinity in finite time, the developed observation procedure shall be valid in large bounded set that can be a priori estimated.

2.1. Circle-criterion-observer design for systems with globally Lipschitz nonlinearities

In this section, we show how to conceive converging observers by employing multiple-output-injection terms. The number of the nonlinear injection terms depends essentially on

[1] It means systems that are discrete in nature.

the number of the positive-slope nonlinearities that are present in the system. This idea permits to exploit each nonlinearity in observer design without making any severe assumption on the whole vector nonlinearity. We show that the proposed design is less conservative as compared with classical design methods especially when the Lipschitz constants are large. The result is summarized in the following statement.

Theorem 1. *Consider system* (1) *satisfying* $(|\mathrm{d}f_i(s)/\mathrm{d}s|)_{1\leqslant i\leqslant\mu}<\infty$ *for all* $s\in\mathbb{R}$. *Let* $(\beta_i)_{1\leqslant i\leqslant\mu}$ *and* $(\varrho_{\min}(i))_{1\leqslant i\leqslant\mu}$ *be two sets of positive constants such that*

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}(f_i(s)+\beta_i s)\right)^{-1}>\varrho_{\min}(i),\quad \forall s\in\mathbb{R},\quad 1\leqslant i\leqslant\mu. \tag{3}$$

If there exist a symmetric and positive definite matrix $P\in\mathbb{R}^{n\times n}$, *a constant matrix* $Y\in\mathbb{R}^{n\times p}$ *and a set of row vectors* $(K_i)_{1\leqslant i\leqslant\mu}\in\mathbb{R}^p$ *such that the following LMIs hold*:

$$(\mathscr{C}_1)\begin{bmatrix}-P & A'P-\sum_{i=1}^{\mu}\beta_i H_i' G_i' P + C'Y' \\ \star & -P\end{bmatrix}<0,$$

$$(\mathscr{C}_2) G_i' P\left(A-\sum_{i=1}^{\mu}\beta_i G_i H_i\right)+G_i' Y C = -\frac{\mu}{2}(H_i+K_iC),\quad 1\leqslant i\leqslant\mu,$$

$$(\mathscr{C}_3) G_i' P G_i - \varrho_{\min}(i)\leqslant 0,\quad 1\leqslant i\leqslant\mu, \tag{4}$$

then, $\lim_{k\to\infty} x_k-\hat{x}_k=0$, *where* $\hat{x}_k$ *is the state vector of the nonlinear discrete-time observer*

$$\hat{x}_{k+1}=A\hat{x}_k+\sum_{i=1}^{\mu}G_i f_i(H_i\hat{x}_k+K_i(C\hat{x}_k-y_k)) + \psi(u_k,y_k)+\sum_{i=1}^{\mu}\beta_i G_i K_i(C\hat{x}_k-y_k) + P^{-1}Y(C\hat{x}_k-y_k). \tag{5}$$

Proof. Let $\mathscr{G}_i(s_k)\triangleq f_i(s_k)+\beta_i s_k,\ 1\leqslant i\leqslant\mu$. Then, system (43) and observer (46) can be rewritten, respectively, as follows:

$$x_{k+1}=\left(A-\sum_{i=1}^{\mu}\beta_i G_i H_i\right)x_k+\sum_{i=1}^{\mu}G_i\mathscr{G}_i(H_i x_k)+\psi(u_k,y_k),$$
$$y_k=Cx_k,\quad (x_k,u_k)\in\Omega\times\mathscr{U},$$

$$\hat{x}_{k+1}=\left(A-\sum_{i=1}^{\mu}\beta_i G_i H_i\right)\hat{x}_k + \sum_{i=1}^{\mu}G_i\mathscr{G}_i(H_i\hat{x}_k+K_i(C\hat{x}-y_k))+\psi(u_k,y_k) + P^{-1}Y(C\hat{x}_k-y_k). \tag{6}$$

Let $A_c\triangleq A-\sum_{i=1}^{\mu}\beta_i G_i H_i$. Then, if we note the observation error as $e_k=x_k-\hat{x}_k$. This implies that

$$e_{k+1}=(A_c+P^{-1}YC)e_k+\sum_{i=1}^{\mu}G_i\mathscr{G}_i(H_i x_k) - \sum_{i=1}^{\mu}G_i\mathscr{G}_i(H_i\hat{x}_k+K_i(C\hat{x}_k-y_k)). \tag{7}$$

Using the mean-value theorem, for a given scalar $\mathscr{C}^{(1)}$-function $\varphi(\cdot)$, we have

$$\varphi(v)-\varphi(w)=\int_0^1 \left.\frac{\partial\varphi(s)}{\partial s}\right|_{s=v-\lambda(v-w)}(v-w)\,\mathrm{d}\lambda.$$

Then, if we put $v_i(k)\triangleq H_i x_k$, $w_i(k)\triangleq H_i\hat{x}_k+K_i(C\hat{x}_k-y_k)$, $\omega_i(k)\triangleq v_i(k)-\lambda(v_i(k)-w_i(k))$, the observation error dynamics can be rewritten as

$$\begin{aligned} e_{k+1}&=(A_c+P^{-1}YC)e_k + \int_0^1\sum_{i=1}^{\mu}G_i\left.\frac{\partial\mathscr{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)}(H_i+K_iC)e_k\,\mathrm{d}\lambda \\ &=\int_0^1(A_c+P^{-1}YC)e_k\,\mathrm{d}\lambda + \int_0^1\sum_{i=1}^{\mu}G_i\left.\frac{\partial\mathscr{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)}(H_i+K_iC)e_k\,\mathrm{d}\lambda. \end{aligned} \tag{8}$$

By taking the Lyapunov function $V_k=e_k'Pe_k$, then we obtain

$$\begin{aligned} V_{k+1}-V_k&=e_{k+1}'Pe_{k+1}-e_k'Pe_k \\ &=\left[\int_0^1(A_c+P^{-1}YC)e_k\,\mathrm{d}\lambda + \int_0^1\sum_{i=1}^{\mu}G_i\left.\frac{\partial\mathscr{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)}(H_i+K_iC)e_k\,\mathrm{d}\lambda\right]' \\ &\quad\times P\times\left[\int_0^1(A_c+P^{-1}YC)e_k\,\mathrm{d}\lambda + \int_0^1\sum_{i=1}^{\mu}G_i\left.\frac{\partial\mathscr{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)}(H_i+K_iC)e_k\,\mathrm{d}\lambda\right] \\ &\quad - e_k'Pe_k. \end{aligned} \tag{9}$$

Using the fact that for given matrix $M=M'>0$, a scalar $\gamma>0$ and vector function $\omega:[0,\ \gamma]\mapsto\mathbb{R}^n$, we have

$$\gamma\int_0^{\gamma}\omega'(\beta)M\omega(\beta)\,\mathrm{d}\beta\geqslant\left(\int_0^{\gamma}\omega(\beta)\,\mathrm{d}\beta\right)'M\left(\int_0^{\gamma}\omega(\beta)\,\mathrm{d}\beta\right),$$

then

$$\Delta V_k \triangleq V_{k+1} - V_k \leqslant \int_0^1 \left[(A_c + P^{-1}YC)e_k + \sum_{i=1}^{\mu} G_i \left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} (H_i + K_i C)e_k \right]' \times P \times \left[(A_c + P^{-1}YC)e_k + \sum_{i=1}^{\mu} G_i \left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} (H_i + K_i C)e_k \right] \mathrm{d}\lambda - \int_0^1 e_k' P e_k \, \mathrm{d}\lambda. \tag{10}$$

By expanding the right-hand side of the last inequality, we obtain

$$\Delta V_k \leqslant \int_0^1 e_k'(A_c + P^{-1}YC)'P(A_c + P^{-1}YC)e_k \, \mathrm{d}\lambda - \int_0^1 e_k' P e_k \, \mathrm{d}\lambda + 2\int_0^1 \sum_{i=1}^{\mu} \left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} \times e_k'(H_i + K_i C)' G_i' P (A_c + P^{-1}YC)e_k \, \mathrm{d}\lambda + \int_0^1 \left[\sum_{i=1}^{\mu} \left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} e_k'(H_i + K_i C)' G_i' \right] P \times \left[\sum_{i=1}^{\mu} \left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} G_i (H_i + K_i C)e_k \right] \mathrm{d}\lambda. \tag{11}$$

By the Cauchy–Schwartz inequality,

$$\mu \sum_{i=1}^{\mu} a_i' P a_i \geqslant \sum_{i=1}^{\mu} a_i' P \left(\sum_{i=1}^{\mu} a_i \right) \quad \text{with } a_i \in \mathbb{R}^n, \quad P \in \mathbb{R}^{n \times n}.$$

Then, we can write that

$$\int_0^1 \left[\sum_{i=1}^{\mu} \left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} e_k'(H_i + K_i C)' G_i' \right] P \times \left[\sum_{i=1}^{\mu} \left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} G_i (H_i + K_i C)e_k \right] \mathrm{d}\lambda \leqslant \mu \int_0^1 \sum_{i=1}^{\mu} \left(\left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} \right)^2 \times e_k'(H_i + K_i C)' G_i' P G_i (H_i + K_i C)e_k \, \mathrm{d}\lambda. \tag{12}$$

This implies that if the following holds:

$$\Delta V_k \leqslant \int_0^1 e_k'(A_c + P^{-1}YC)'P(A_c + P^{-1}YC)e_k \, \mathrm{d}\lambda - \int_0^1 e_k' P e_k \, \mathrm{d}\lambda + 2\int_0^1 \sum_{i=1}^{\mu} \left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} \times e_k'(H_i + K_i C)' G_i' P (A_c + P^{-1}YC)e_k \, \mathrm{d}\lambda + \mu \int_0^1 \sum_{i=1}^{\mu} \left(\left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} \right)^2 e_k'(H_i + K_i C)' \times G_i' P G_i (H_i + K_i C)e_k \, \mathrm{d}\lambda \tag{13}$$

then (11) holds. Let us choose P such that

$$(A_c + P^{-1}YC)'P(A_c + P^{-1}YC) - P = -Q < 0, \quad Q > 0, \tag{14}$$

or, equivalently (by the Schur complement),

$$\begin{bmatrix} -P & A'P - \sum_{i=1}^{\mu} \beta_i H_i' G_i' P + C'Y' \\ \star & -P \end{bmatrix} < 0. \tag{15}$$

By adding the following equality constraints:

$$G_i' P A_c + G_i' Y C = -\frac{\mu}{2}(H_i + K_i C), \quad 1 \leqslant i \leqslant \mu, \tag{16}$$

then we obtain

$$\Delta V_k \leqslant -e_k' Q e_k - \mu \int_0^1 \sum_{i=1}^{\mu} \left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} \times e_k'(H_i + K_i C)'(H_i + K_i C)e_k \, \mathrm{d}\lambda + \mu \int_0^1 \sum_{i=1}^{\mu} \left(\left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} \right)^2 \times e_k'(H_i + K_i C)' G_i' P G_i (H_i + K_i C)e_k \, \mathrm{d}\lambda. \tag{17}$$

Since $\varrho_{\min}(i) < (\partial \mathcal{G}_i(s_k)/\partial s_k|_{s_k=\omega_i(k)})^{-1}$, $1 \leqslant i \leqslant \mu$, and $G_i' P G_i - \varrho_{\min}(i) \leqslant 0$, $1 \leqslant i \leqslant \mu$, then

$$\Delta V_k \leqslant -e_k' Q e_k - \mu \int_0^1 \sum_{i=1}^{\mu} \left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} e_k'(H_i + K_i C)' \times \left[1 - G_i' P G_i \left.\frac{\partial \mathcal{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} \right] (H_i + K_i C)e_k \, \mathrm{d}\lambda \leqslant -e_k' Q e_k \leqslant 0. \tag{18}$$

Since $Q > 0$ then the observation error is exponentially stable. This ends the proof. □

Remark 1. For globally Lipschitz systems where $|\mathrm{d} f_i(s)/\mathrm{d}s| < \infty$, $1 \leqslant i \leqslant \mu$, $\forall s \in \mathbb{R}$, condition (2) is not necessary for the observer design. Condition (3) suffices for the determination of the coefficients $(\beta_i)_{1\leqslant i\leqslant \mu}$.

It is worth to mention that from Eq. (8), the observation error dynamics can be rewritten as

$$e_{k+1} = (A_c + P^{-1}YC)e_k + \sum_{i=1}^{\mu} G_i\varphi_i(k, z_i(k)),$$
$$z_i(k) = (H_i + K_iC)\, e_k, \tag{19}$$

where $A_c + P^{-1}YC$ is a stable matrix, and

$$\varphi_i(k, z_i(k)) \triangleq \int_0^1 \left.\frac{\partial \mathscr{G}_i(s_k)}{\partial s_k}\right|_{s_k=\omega_i(k)} z_i(k)\,\mathrm{d}\lambda.$$

According to (19), the observer design problem is equivalent to a stabilization of a linear discrete-time system interconnected with a sum of memoryless nonlinearities verifying the sector conditions $z_i\varphi_i(k, z_i(k))\geqslant 0$. In this subsection, we highlight the connection between the LMIs conditions of the main Theorem 1 and the positive realness of the transfer functions

$$\boldsymbol{G}_i(z) \overset{\min}{\sim} \left[\begin{array}{c|c} A_c + P^{-1}YC & G_i \\ \hline -\frac{\mu}{2}\left(H_i + K_iC\right) & \frac{1}{2}\varrho_{\min}(i)\end{array}\right], \quad 1\leqslant i\leqslant \mu, \tag{20}$$

where $\overset{\min}{\sim}$ designates the minimal realization. To summarize, we show the following corollary.

Corollary 1. *Consider system* (1) *and observer* (5). *If all the conditions of Theorem* 1 *hold then, for* $1\leqslant i\leqslant\mu$, *the transfer functions* $\boldsymbol{G}_i(z)$ *defined as*

$$-\frac{\mu}{2}(H_i + K_iC)\left(zI - A + \sum_{i=1}^{\mu}\beta_i G_i H_i - P^{-1}YC\right)^{-1} G_i$$
$$+\frac{1}{2}\varrho_{\min}(i) \tag{21}$$

are strongly positive real.

Proof. By the use of Lemma 4.2 given in Haddad and Bernstein (1994), we can write that $(\boldsymbol{G}_i(z))_{1\leqslant i\leqslant\mu}$ are strongly positive real if the following matrix inequalities hold for $1\leqslant i\leqslant\mu$:

$$\begin{bmatrix} A'_{\text{closed}}PA_{\text{closed}} - P & \star \\ -\frac{\mu}{2}(H_i + K_iC) - G'_iPA_{\text{closed}} & -\varrho_{\min}(i) + G'_iPG_i\end{bmatrix} < 0, \tag{22}$$

where $A_{\text{closed}} \triangleq A_c + P^{-1}YC$. If the conditions of Theorem 1 hold then, for $1\leqslant i\leqslant\mu$, the last matrix inequalities are equivalent to

$$\begin{bmatrix} A'_{\text{closed}}PA_{\text{closed}} - P & \mathbf{0} \\ \mathbf{0} & -\varrho_{\min}(i) + G'_iPG_i\end{bmatrix} < 0 \tag{23}$$

which always hold under the conditions of Theorem 1. □

2.2. Discussion

Condition ($\mathscr{C}_1$) of Theorem 1 is a necessary condition that guarantees the stability of the linear part of the observation error. This condition is always feasible if the pair (A, C) is observable. Condition ($\mathscr{C}_2$) of Theorem 1 is a set of equality constraints from which the gains $(K_i)_{1\leqslant i\leqslant\mu}$ of the nonlinear-output-injection terms are determined. The feasibility of ($\mathscr{C}_2$) is related to the positive realness of the transfer functions $(\boldsymbol{G}_i(z))_{1\leqslant i\leqslant\mu}$ as shown in the statement of Corollary 1. Condition ($\mathscr{C}_3$) is an additional constraint that links the matrix P to the slope of nonlinearities $(\mathscr{G}_i(s))_{1\leqslant i\leqslant\mu}$. Remark that the slope information (or the Lipschitz constants) does not appear in condition ($\mathscr{C}_1$). However, depending upon the number of nonlinearities that are present in the system dynamics, the restriction of the slopes are translated as algebraic constraints on one or more elements of the matrix P, see condition ($\mathscr{C}_3$). As a comparison with the results given in Richter and Misawa (2003), conditions ($\mathscr{C}_1$) and ($\mathscr{C}_3$) are not given as equality constraints. Furthermore, condition ($\mathscr{C}_2$) permits the isolation of the observer output injection terms so as to deal with different kinds of nonlinearities.

According to conditions ($\mathscr{C}_1$) and ($\mathscr{C}_3$), we stress that the memoryless nonlinearities are not considered as linear perturbations terms that can be associated to the matrix A, and, hence, more flexibility is offered by the design method. When the sampling rate is sufficiently small, condition ($\mathscr{C}_3$) can be neglected since the term of G'_iPG_i in (17) involves implicitly the square of the sampling period. Indeed, the conditions become free from the slopes of nonlinearities.

In order to motivate and compare our work with classical Luenberger observer design for globally Lipschitz systems, let us replace the nonlinearities in (1) by a single vector $f(x_k)$. Then, (1) takes the form

$$\begin{cases} x_{k+1} = Ax_k + f(x_k) + \psi(u_k, y_k), \\ y_k = Cx_k. \end{cases} \tag{24}$$

Since we assume that any information about $f(x_k)$ is available, then, the standard Lipschitz property is considered. That is, there exists $G\in\mathbb{R}^{n\times n}$ such that for $x_k, \hat{x}_k \in \Omega\subseteq\mathbb{R}^n$

$$\|f(x_k) - f(\hat{x}_k)\| \leqslant \|G(x_k - \hat{x}_k)\|. \tag{25}$$

The high-gain observer is readily constructed as

$$\begin{cases} \hat{x}_{k+1} = A\hat{x}_k + f(\hat{x}_k) + \psi(u_k, y_k) + X^{-1}Z(\hat{y}_k - y_k), \\ \hat{y}_k = C\hat{x}_k. \end{cases} \tag{26}$$

Setting $e_k = \hat{x}_k - x_k$ and $V_k = e'_kXe_k$, then we can prove by the use of the S-procedure lemma that the following dynamics:

$$e_{k+1} = (A + X^{-1}ZC)e_k + f(\hat{x}_k) - f(x_k) \tag{27}$$

is stable, or $V_{k+1} - V_k < 0$ under (25) if and only if there exists $\varepsilon > 0$ such that the following holds:

$$\begin{bmatrix} -X & A'X + C'Z' & \varepsilon G' & A'X + C'Z' \\ \star & -X & \mathbf{0} & \mathbf{0} \\ \star & \star & -\varepsilon I & \mathbf{0} \\ \star & \star & \star & X - \varepsilon I \end{bmatrix} < 0. \tag{28}$$

Since the observation error dynamics can be rewritten as

$$e_{k+1} = \int_0^1 (A + \Delta A(\lambda, k) + X^{-1} ZC) e_k \, \mathrm{d}\lambda, \tag{29}$$

where $\Delta A(\lambda, k) = \partial f(s_k)/\partial s_k|_{s_k = \hat{x}_k + \lambda(x_k - \hat{x}_k)}$ is a bounded uncertainty, then, we would obtain the same condition (28) if we consider the following stabilization of the error dynamics:

$$e_{k+1} = (A + \Delta A(\lambda, k)) e_k + L\, \tilde{y}_k,$$
$$\tilde{y}_k = C e_k, \tag{30}$$

where the gain $L = X^{-1} Z$ is computed so as to guarantee the *quadratic stability* of the linear uncertain system (30). Referring to the abundant literature that is devoted to the stability of uncertain linear systems with output feedback, the problem of how much uncertainty can be dealt with by a static output feedback is generally raised, see e.g., Xie and Guo (2000) and Oliveira, Bernussou, and Geromel (1999). As a result, there is always a limitation which restricts the design of Luenberger observers with a unique linear-output-injection term even if the norm of the uncertainty $\Delta A(\lambda, k)$ (or the Lipschitz constant) is small. If we see attentively to the developed condition (28), we realize that the conservatism of this condition is mainly due to the fact that only one output injection term $L(C\hat{x} - Cx)$ is used to compensate the adverse nonlinearity $f(\hat{x}_k) - f(x_k)$. In addition, the poor knowledge of the nonlinearity $f(x_k)$ has also led to the conservative condition (25) that bounds the difference $f(\hat{x}_k) - f(x_k)$ by its norm. By returning back to the design of the circle-criterion observer, we realize that the design is essentially based on the knowledge of the sign of the nonlinearity slope. This important information permits to remove the usual conservative condition (25). Moreover, the observer design method consists of designing multiple-output-injection terms that offer more freedom in setting up globally converging observers with high Lipschitz constants.

Example 1. Consider the continuous-time nonlinear system

$$\dot{x}_1(t) = x_2(t) + \frac{\gamma}{2} \sin(x_1(t) + x_2(t)) + u(t),$$
$$\dot{x}_2(t) = \gamma \sin(x_1(t) + x_2(t)) + u(t),$$
$$y(t) = x_1(t), \tag{31}$$

where γ is a positive real constant. The Lipschitz constant of the aforementioned system is equal to 2γ. By taking the Euler discrete-time model with sampling period δ, we obtain

$$x_{k+1} = Ax_k + \delta\gamma G_1 \sin(H_1\, x_k) + \delta B u_k,$$
$$y_k = Cx_k, \tag{32}$$

Table 1

γ	P	Y	K_1
1	$\begin{bmatrix} 62.421 & -13.249 \\ -13.249 & 6.0711 \end{bmatrix}$	$\begin{bmatrix} -36.07 \\ 0.63377 \end{bmatrix}$	−1.8687
5	$\begin{bmatrix} 21.891 & -4.5643 \\ -4.5643 & 1.9313 \end{bmatrix}$	$\begin{bmatrix} -10.841 \\ 0.083439 \end{bmatrix}$	−2.663
10	$\begin{bmatrix} 22.168 & -5.3463 \\ -5.3463 & 2.5237 \end{bmatrix}$	$\begin{bmatrix} -9.1738 \\ 0.0054277 \end{bmatrix}$	−2.4967
50	$\begin{bmatrix} 48.486 & -22.881 \\ -22.881 & 11.429 \end{bmatrix}$	$\begin{bmatrix} -0.56831 \\ 0.3474 \end{bmatrix}$	−2.8462
5000	$\begin{bmatrix} 3898.5 & -1949.3 \\ -1949.3 & 974.7 \end{bmatrix}$	$\begin{bmatrix} -92.064 \\ 47.633 \end{bmatrix}$	−3.0065
40,000	$\begin{bmatrix} 33,885 & -16,943 \\ -16,943 & 8471.3 \end{bmatrix}$	$\begin{bmatrix} -648.94 \\ 325.95 \end{bmatrix}$	−3.0015

where

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = (I + \delta F), \quad G_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \quad C = [1\ 0],$$
$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}'. \tag{33}$$

In order to apply the result of Theorem 1, let us rewrite the dynamics of system (32) as follows:

$$x_{k+1} = A_c x_k + G_1[\delta\gamma \sin(H_1 x_k) + \beta_1 H_1 x_k] + \delta B u_k,$$
$$y_k = Cx_k, \tag{34}$$

where $A_c = A - \beta G_1 H_1 = \begin{bmatrix} 1 - \frac{1}{2}\beta_1 & \delta - \frac{1}{2}\beta_1 \\ -\beta_1 & 1 - \beta_1 \end{bmatrix}$ and $\beta_1 \triangleq \frac{3}{2}\delta\gamma$, $\mathscr{G}_1(s) \triangleq \delta\gamma[\sin(s) + \frac{3}{2}s]$. Here, $\mathrm{d}\mathscr{G}_1(s)/\mathrm{d}s = \delta\gamma[\cos(s) + \frac{3}{2}] > 0$. Consequently, we can choose $\varrho_{\min} = 2/5\delta\gamma$. The objective of introducing this example is to show that the LMIs of Theorem 1 are not conservative when the value of γ increases. Therefore, we shall check the solvability of LMIs (4) for increasing values of γ. The results are given in Table 1 for $\delta = 0.01$. Hence, the states of the following observer:

$$\begin{aligned} \hat{x}_{k+1} = {} & A_c \hat{x}_k + G_1[\delta\gamma \sin(H_1 \hat{x}_k + K_1(C\hat{x}_k - y_k)) \\ & + \beta_1(H_1 \hat{x}_k + K_1(C\hat{x}_k - y_k))] + \delta B u_k \\ & + P^{-1} Y (C\hat{x}_k - y_k) \end{aligned}$$

converge asymptotically to the states of system (32) for any initial conditions $\hat{x}_0$. Note that the observer gains K_1 and $P^{-1}Y$ are not high-gain vectors even if γ increases. This nice property has undoubtedly an impact on the robustness of the transient behavior of the observer.

2.3. Comparisons

Now, let us compare the conservativeness of both LMI (28) and LMIs of Theorem 1. In order to check the feasibility of LMI (28), we rewrite system (32) as follows:

$$x_{k+1} = \underbrace{\begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} \frac{1}{2}\delta\gamma\sin(x_1(k)+x_2(k)) \\ \delta\gamma\sin(x_1(k)+x_2(k)) \end{bmatrix}}_{f(x_k)} + \underbrace{\begin{bmatrix} \delta u_k \\ \delta u_k \end{bmatrix}}_{\psi(u_k,y_k)},$$

$$y_k = \underbrace{[1\ 0]}_{C} x_k. \tag{35}$$

According to the last equations, we fix $G=\delta\gamma\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ 1 & 1\end{bmatrix}$. Notice that LMI (28) is solvable for $\delta=0.01$ with $1\leqslant\gamma\leqslant 0.89$. When $\gamma>1$, LMI (28) becomes unsolvable. As we have recorded in Table 1, LMIs of Theorem 1 remain solvable even for $\gamma=4\times10^4$ (with a maximum error of order 10^{-8} on equality constraints). Thus, we can say that the Lipschitz constant of the continuous-time system (31) reaches the value of $800/\delta=2\gamma$ which is a very important limit as compared with previous results (Ren & Guo, 2005; Xue & Guo, 2002). As we have mentioned in the Discussion section, the stability of the observation error is equivalent to the stability condition of a linear uncertain system with structured uncertainties, see (30). The algorithm presented by Oliveira et al. in Oliveira et al. (1999) showed that the design provides better results than the classical quadratic stability. For the sake of comparison, let us consider again system (35). The Jacobian of $f(x_k)$ can be rewritten as polytopic uncertainties, that is, $\partial f(x_k)/\partial x_k \in \mathscr{A}(\delta\gamma) \triangleq \mathrm{Co}\left\{\pm\delta\gamma\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ 1 & 1\end{bmatrix}\right\}$, where "Co" designates the convex hull of matrices. To apply the result given in Oliveira et al. (1999, Theorem 2, LMI (7)), define P_1, P_2, Y and G as LMIs variables and set $A_1=A+\delta\gamma\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ 1 & 1\end{bmatrix}+G^{-1}YC$ and $A_2=A-\delta\gamma\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ 1 & 1\end{bmatrix}+G^{-1}YC$. By solving LMI (7) of Oliveira et al. (1999), we found that we could get solutions for δ=0.01 and $0\leqslant\gamma\leqslant 66$. However, for $\gamma>67$, the design proposed in Oliveira et al. (1999) becomes unfeasible. Now, let us focus on the example studied in Rajamani (1998, Section V). In order to be in the same conditions of this example, we shall modify the matrix F and C in (32) by $\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}$ and $[0\quad 1]$, respectively. The rest of the nominal matrices and coefficients are the same as in (32). In this example (see Rajamani, 1998, Section V) it is showed that the Lipschitz constant cannot exceed the value 0.49. Even though our analysis is carried out in discrete time where the sampling period $\delta=0.01$ appears as an additional constraint, the LMIs of Theorem 1 are solvable until $\gamma=2\times10^4$ where $P=\begin{bmatrix}15{,}985 & -7992.3\\ -7992.3 & 3996.1\end{bmatrix}$, $Y=\begin{bmatrix}-182.9\\ 92.253\end{bmatrix}$ and $K_1=-1.5015$. For this solution, we record a maximum error of order 10^{-8} on equality constraints. Indeed, the proposed method allows the Lipschitz constant of the nonlinearity to reach the value 4×10^4 which is a very high value as compared with 0.49 obtained in Rajamani (1998).

3. Extension to bounded-state nonlinear systems

In this section, we consider the observation problem of discrete-time nonlinear systems whose states live in a large compact set. The main objective of this Section is to extend the circle-criterion-observer design to bounded-state systems whose nonlinearities can be seen as globally Lipschitz in a large compact set that belongs to $\mathbb{R}^n$. Before giving the main result of this paper, we begin by exposing the following important result.

Lemma 1. *Consider the saturation function $\mathscr{S}(v)$ defined as*

$$\mathscr{S}(v) \triangleq \begin{cases} v & \text{if } -\rho\leqslant v\leqslant\rho, \\ \rho+(v-\rho)\mathrm{e}^{\rho-v} & \text{if } v>\rho, \\ -\rho+(v+\rho)\mathrm{e}^{\rho+v} & \text{if } v<-\rho. \end{cases} \tag{36}$$

Then, $\mathscr{S}(v)$ and $\mathrm{d}\mathscr{S}(v)/\mathrm{d}v$ are bounded and continuous over $\mathbb{R}$.

Proof. The function $\mathscr{S}(v)$ can be rewritten as follows:

$$\mathscr{S}(v) \triangleq \begin{cases} v \quad \text{if } |v|\leqslant\rho, \\ \rho\,\mathrm{sign}\,(v-\rho)+(v-\rho\,\mathrm{sign}\,(v-\rho))\mathrm{e}^{\rho-|v|} \\ \qquad \text{if } |v|>\rho, \end{cases} \tag{37}$$

where sign(·) designates the habitual sign function. We have

$$\begin{aligned}\lim_{v\to\pm\rho} v &= \lim_{v\to\pm\rho} \pm\rho+(v\mp\rho)\,\mathrm{e}^{\rho-|v|} \\ &= \pm\rho. \end{aligned} \tag{38}$$

This implies that $\mathscr{S}(v)$ is continuous. Similarly, its first derivative with respect to v is defined as

$$\frac{\mathrm{d}}{\mathrm{d}v}\mathscr{S}(v) \triangleq \begin{cases} 1 & \text{if } -\rho\leqslant v\leqslant\rho, \\ \mathrm{e}^{\rho-v}(1-v+\rho) & \text{if } v>\rho, \\ \mathrm{e}^{\rho+v}(1+v+\rho) & \text{if } v<-\rho. \end{cases} \tag{39}$$

According to the last definition, we have

$$\lim_{v\to\rho} \mathrm{e}^{\rho-v}(1-v+\rho) = \lim_{v\to-\rho} \mathrm{e}^{\rho+v}(1+v+\rho) = 1. \tag{40}$$

Then, we conclude that the function $\mathrm{d}\mathscr{S}(v)/\mathrm{d}v$ is also continuous. In order to prove the boundedness of $\mathscr{S}(v)$ and $\mathrm{d}\mathscr{S}(v)/\mathrm{d}v$, it is sufficient to see from definitions (37) and (39) that $\lim_{|v|\to\infty}(v-\rho\,\mathrm{sign}\,(v-\rho))\mathrm{e}^{\rho-|v|}=0$, and $\lim_{|v|\to\infty}\mathrm{e}^{\rho-|v|}(1-|v|+\rho)=0$. This ends the proof. □

Consider now system (1) where all the system states are assumed to be bounded for given initial condition $x_0\in\Omega\subset\mathbb{R}^n$ and bounded input $u_k\in\mathscr{U}\subset\mathbb{R}^m$. Using the result of Lemma 1, we can always find a set of positive constants $(\rho_i)_{1\leqslant i\leqslant\mu}$ and a set of real numbers $(\tau_i)_{1\leqslant i\leqslant\mu}$ such that

$$\begin{aligned} &f_i(H_i x_k)=\mathscr{S}_i\circ f_i(H_i x_k), \quad 1\leqslant i\leqslant\mu,\ \ x_k\in\Omega, \\ &f_i(\tau_i)=\rho_i, \quad 1\leqslant i\leqslant\mu, \end{aligned} \tag{41}$$

where

$$\mathscr{S}_i(v) \triangleq \begin{cases} v & \text{if } -\rho_i \leqslant v \leqslant \rho_i, \\ \rho_i + (v-\rho_i)\mathrm{e}^{\rho_i - v} & \text{if } v > \rho_i, \\ -\rho_i + (v+\rho_i)\mathrm{e}^{\rho_i + v} & \text{if } v < -\rho_i. \end{cases} \quad (42)$$

Consequently, system (1) can be rewritten in the following form:

$$x_{k+1} = Ax_k + \sum_{i=1}^{\mu} G_i \mathscr{S}_i \circ f_i(H_i x_k) + \psi(u_k, y_k),$$

$$y_k = Cx_k, \quad (x_k, u_k) \in \Omega \times \mathscr{U}. \quad (43)$$

Due to the developed saturation functions $(\mathscr{S}_i(v))_{1\leqslant i\leqslant \mu}$, the bounded-state system (1) is viewed as a *smooth dynamical system* with bounded nonlinearities. The employed saturation functions $(\mathscr{S}_i(v))_{1\leqslant i\leqslant \mu}$ approach the classical non-differentiable saturation functions

$$\mathrm{Sat}_i(v) \triangleq \begin{cases} v & \text{if } -\rho_i \leqslant v \leqslant \rho_i, \\ \rho_i & \text{if } v > \rho_i, \\ -\rho_i & \text{if } v < -\rho_i \end{cases} \quad (44)$$

when $|v| \gg (\rho_i)_{1\leqslant i\leqslant \mu}$. In this section, the new equivalent structure of system (43) is exploited to build converging observers that enjoy the properties to be smooth too. The design of the observer is given by the following statement.

Corollary 2. *Consider system* (1) *under assumption* (2). *Define* $\Omega \triangleq \{x_k \in \mathbb{R}^n \mid |x_i(k)| \leqslant \alpha_i,\ 1\leqslant i\leqslant n\}$ *with* $\alpha_i > 0, 1\leqslant i\leqslant n$. *Assume that for all bounded input* $u_k \in \mathscr{U} \subset \mathbb{R}^m$, *and some initial conditions* $x_0 \in \Omega$, *the state vector* x_k *belongs to the same subset* Ω *for all* $k \in \mathbb{Z}_{>0}$. *Let* $(\rho_i)_{1\leqslant i\leqslant \mu}$ *be positive saturation levels defined as in* (41)–(42). *If we choose two sets of positive constants* $(\beta_i)_{1\leqslant i\leqslant \mu}$ *and* $(\varrho_{\min}(i))_{1\leqslant i\leqslant \mu}$ *such that for* $1\leqslant i\leqslant \mu$

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}(\mathscr{S}_i \circ f_i(s) + \beta_i s)\right)^{-1} > \varrho_{\min}(i), \quad \forall s \in \mathbb{R}, \quad (45)$$

and there exist a symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$, *a constant matrix* $Y \in \mathbb{R}^{n\times p}$ *and a set of row vectors* $(K_i)_{1\leqslant i\leqslant \mu} \in \mathbb{R}^p$ *such that the conditions* $(\mathscr{C}_1)$–$(\mathscr{C}_3)$ *of Theorem* 1 *hold, then for any initial condition* $\hat{x}_0$, *the states of the following observer*:

$$\begin{aligned} \hat{x}_{k+1} = {} & A\hat{x}_k + \sum_{i=1}^{\mu} G_i \mathscr{S}_i \circ f_i(H_i \hat{x}_k + K_i(C\hat{x}_k - y_k)) \\ & + \psi(u_k, y_k) + \sum_{i=1}^{\mu} \beta_i G_i K_i (C\hat{x}_k - y_k) \\ & + P^{-1} Y (C\hat{x}_k - y_k) \end{aligned} \quad (46)$$

converge asymptotically to the states of system (1).

Proof. The proof is omitted here because it is quite similar to the proof of Theorem 1. The only difference in the proof is that $\mathscr{G}_i(s_k)$ becomes equal to $\mathscr{S}_i(s_k) \circ f_i(s_k) + \beta_i s_k$. This ends the proof. □

Remark 2. A practical method to determine the coefficient $(\beta_i)_{1\leqslant i\leqslant \mu}$ for a given saturation level $(\rho_i)_{1\leqslant i\leqslant \mu}$ is to see by how much the functions $g_i(s) \triangleq (\mathrm{d}f_i(s)/\mathrm{d}s)\mathrm{e}^{\rho_i - |f_i(s)|}(1 - |f_i(s)| + \rho_i)$, $|f_i(s)| > \rho_i$, $1\leqslant i\leqslant \mu$, drop below zero. The coefficients $(\beta_i)_{1\leqslant i\leqslant \mu}$ are determined as the minimum values that make $g_i(s) + \beta_i > 0$ for all i.

Note that from the LMI conditions of Corollary 2, we realize that, if the matrix P verifies the conditions $G_i' P G_i = 0$ for all i, then the breakdown of the observer will be independent from the slopes of nonlinearities. However, if the conditions $G_i' P G_i = 0$, $1\leqslant i\leqslant \mu$, are imposed, the positive definite requirement of P should be weakened to positive semi-definite. As a result, the linear-output-injection term cannot be computed through $P^{-1}Y$ since P may not be invertible. For this particular reason, these conditions are not considered herein. However, the conditions $G_i' P G_i \leqslant \varepsilon_i$, $1\leqslant i\leqslant \mu$, can be imposed for small values of ε_i which means that the slopes of nonlinearities are maximized. Hence, the domain of observation can be set as large as possible.

Corollary 3. *Consider system* (1). *If there exist a symmetric and positive definite matrix* $P \in \mathbb{R}^{n\times n}$, *a constant matrix* $Y \in \mathbb{R}^{n\times p}$, *a set of row vectors* $(K_i)_{1\leqslant i\leqslant \mu} \in \mathbb{R}^p$ *and a set of positive constants* $(\varepsilon_i)_{1\leqslant i\leqslant \mu}$ *such that the following optimization problem is solvable*:

$$\min_{P, Y, (K_i)_{1\leqslant i\leqslant \mu}} \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\mu,$$

subject to

$$\begin{bmatrix} -P & A'P - \sum_{i=1}^{\mu} \beta_i H_i' G_i' P + C'Y' \\ \star & -P \end{bmatrix} < 0,$$

$$G_i' P \left(A - \sum_{i=1}^{\mu} \beta_i G_i H_i\right) + G_i' Y C = -\frac{\mu}{2}(H_i + K_i C), \quad 1\leqslant i\leqslant \mu,$$

$$G_i' P G_i - \varepsilon_i \leqslant 0, \quad 1\leqslant i\leqslant \mu, \quad (47)$$

then there exist a set of saturation levels $(\rho_i)_{1\leqslant i\leqslant \mu} > 0$ *and a set* $(\beta_i)_{1\leqslant i\leqslant \mu} > 0$ *such that* $(\mathrm{d}/\mathrm{d}s)(\mathscr{S}_i \circ f_i(s) + \beta_i s) \leqslant \varepsilon_i^{-1}$ *for all i and, consequently, the states of observer* (46) *converge asymptotically to those of system* (1) *whenever the states of system* (1) *do not leave the set* $\mathscr{D}$ *defined as*

$$\mathscr{D} = \left\{ x_k \in \mathbb{R}^n \,\middle|\, |H_i x_k| \leqslant \left(\frac{\mathrm{d}\mathscr{G}_i(s)}{\mathrm{d}s}\right)^{(-1)}\Bigg|_{s=\frac{1}{\varepsilon_i}}, \quad 1\leqslant i\leqslant \mu, \quad k \in \mathbb{Z}_{\geqslant 0} \right\},$$

where $\mathscr{G}_i(s) = \mathscr{S}_i \circ f_i(s) + \beta_i s\ \forall i$.

Proof. The result of this corollary is a direct consequence of Corollary 2. The result of Corollary 3 shows the inverse design of Corollary 2 when the slopes of nonlinearities are put as LMIs variables. □

Example 2. In order to show that the presented algorithm can deal with positive- and non-positive slope nonlinearities, let us consider the following nonlinear system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 10 & 0 \\ -10 & 0 & 5 \\ 0 & -\frac{10}{3} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x_3^2(t) \\ x_3^3(t) \end{bmatrix}}_{f(x(t))} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \tag{48}$$

where $y_1(t) = x_1(t)$ and $y_2(t) = x_2(t) + x_3(t)$ are the system outputs. The system nonlinearity $f(x(t))$ does not verify the condition $(\partial f(x(t))/\partial x(t))' + (\partial f(x(t))/\partial x(t)) \geqslant 0$. Therefore, the design proposed in Fan and Arcak (2003) cannot be applied. The Euler discrete-time approximation of the last system gives

$$x_{k+1} = \underbrace{\left(I + \delta \begin{bmatrix} 0 & 10 & -1 \\ -10 & 0 & 5 \\ 0 & -\frac{10}{3} & 0 \end{bmatrix} \right)}_{A} x_k + \delta \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{G_1} f_1(x_3(k)) + \delta \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{G_2} f_2(x_3(k)),$$

$$y_k = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{C} x_k,$$

where $f_1(s) = 2s^3 + s^2 + s$, $f_2(s) = s^3$. Here, the system nonlinearity $x_3^2(k)$ has not always a positive slope. However, by expanding the square nonlinearity $x_3^2(k)$ as follows: $x_3^2(k) = f_1(x_3(k)) - 2f_2(x_3(k)) - f_3(x_3(k))$ where $f_3(s) = s$, it is clear that $f_1(s)$, $f_2(s)$ and $f_3(s)$ have all positive slopes for all $s \in \mathbb{R}$. In the aforementioned system, the function $f_3(s)$ is added to the linear dynamics and $H_1 = H_2 = [0\ 0\ 1]$. For $\delta = 0.05$, we have found that LMIs (47) are feasible, where $\varepsilon_1 = 3 \times 10^{-5}$, $\varepsilon_2 = 3 \times 10^{-4}$,

$$P = \begin{bmatrix} \frac{3}{107,054} & \frac{4571}{3428} & \frac{9}{118,877} \\ \frac{4571}{3428} & \frac{440,863}{6} & \frac{6857}{1714} \\ \frac{9}{118,877} & \frac{6857}{1714} & \frac{21}{51,131} \end{bmatrix},$$

$$Y = \begin{bmatrix} \frac{6147}{9220} & -\frac{2198}{6593} \\ \frac{183,691}{5} & -\frac{73,481}{4} \\ \frac{7651}{3825} & -\frac{2155}{2154} \end{bmatrix},$$

$$K_1 = [-\tfrac{9}{578,006} \quad -\tfrac{20,816}{20,815}],$$

$$K_2 = [-\tfrac{1}{69,868} \quad -\tfrac{132,853}{132,854}]. \tag{49}$$

Example 3. Consider the discrete-time nonlinear system

$$x_{k+1} = \underbrace{\begin{bmatrix} 1 & 0.01 & 0 \\ -0.4860 & -0.9875 & 0.4860 \\ 0 & -0.1 & 1 \end{bmatrix}}_{A} x_k + \underbrace{\begin{bmatrix} 0.2 \\ 0.5 \\ 0.4 \end{bmatrix} u_k}_{\psi(u_k, y_k)} + \delta \underbrace{\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}}_{G_1} \mathrm{e}^{\overbrace{[0\ -1\ -1]}^{H_1} x_k} + \delta \underbrace{\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}}_{G_2} (\underbrace{[0\ 0\ 1]}_{H_2} x_k)^3(k),$$

$$y_k = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}}_{C} x_k, \tag{50}$$

where $x_0 = [-2\ 0\ 0]$ and $\delta = 0.01$ is the sampling period. When we apply the input $u_k = 0.3\sin(3t_k)$ for all k, the trajectories of the state vector are bounded, whereby $|x_1(k)| < 15 = \alpha_1$, $|x_2(k)| < 3 = \alpha_2$, $|x_3(k)| < 3.5 = \alpha_3$. The nonlinearities $f_1(s_k) = \delta \mathrm{e}^{s_k}$ and $f_2(s_k) = \delta s_k^3$ are strictly monotone. According to the upper bounds $(\alpha_i)_{1 \leqslant i \leqslant 3}$, let us fix the saturation levels of nonlinearities as $\rho_1 = \delta \mathrm{e}^{13/2}$, $\rho_2 = \delta(\frac{7}{2})^3$. To fix the values of β_1 and β_2, we plot the evolutions of the functions $\mathrm{d}(\mathscr{S}_1 \circ f_1(s))/\mathrm{d}s$ and $\mathrm{d}(\mathscr{S}_2 \circ f_2(s))/\mathrm{d}s$ in order to see by how much the functions drop below zero when s increases. Then, by fixing $\beta_1 = \frac{3}{2}$, $\beta_2 = 0.2$, the nonlinearities $\mathrm{d}\mathscr{G}_1(s)/\mathrm{d}s$ and $\mathrm{d}\mathscr{G}_2(s)/\mathrm{d}s$ are strictly positive. According to these parameters, we can consequently fix $\varrho_{\min}(1) = 0.1227$ and $\varrho_{\min}(2) = 1.7621$. By the use of the LMI package of Matlab with Sedumi interface, we get a solution of the constrained LMIs(4), that is,

$$P = \begin{bmatrix} 4.3917 & 3.6094 & -3.8167 \\ 3.6094 & 3.2148 & -3.0554 \\ -3.8167 & -3.0554 & 3.3598 \end{bmatrix},$$

$$Y = \begin{bmatrix} -4.6302 & 2.3354 \\ -4.0362 & 1.9971 \\ 3.9304 & -1.9955 \end{bmatrix},$$

$$K_1 = [-0.39396\ \ 0.66726],$$

$$K_2 = [-0.39049 \quad -0.33387]. \tag{51}$$

4. Conclusion

Circle-criterion observers for discrete-time nonlinear systems with both positive- and non-positive-slope nonlinearities are developed. We showed that the existence of such observers is conditioned by the solutions of a set of LMI conditions with equality constraints. The present work can be seen as an extension of existing works on discrete-time Luenberger observers and a counterpart of the work on the circle-criterion-observer design developed in the continuous-time case.

References

Aboky, C., Sallet, G., & Vivalda, L.-C. (2002). Observers for Lipschitz nonlinear systems. *International Journal of Control*, *75*(3), 204–212.

Arcak, M., & Kokotović, P. (2001). Observer-based control of systems with slope-restricted nonlinearities. *IEEE Transactions on Automatic Control*, *46*(7), 1146–1150.

Arcak, M., & Nešić, D. (2004). A framework for nonlinear sampled-data observer design via approximate discrete-time models and emulation. *Automatica*, *40*(11), 1931–1938.

Bestle, D., & Zeitz, M. (1983). Canonical form observer design for nonlinear time variable systems. *International Journal of Control*, *38*(2), 419–431.

Califano, C., Monaco, S., & Normand-Cyrot, D. (2003). On the observer design in discrete-time. *Systems and Control Letters*, *49*(4), 255–265.

Ciccarella, G., Mora, M. D., & Germani, A. (1993). Observers for discrete-time nonlinear systems. *Systems and Control Letters*, *20*, 373–382.

Dabroom, A. M., & Khalil, H. K. (1999). Discrete-time implementation of high-gain observers for numerical differentiation. *International Journal of Control*, *72*(17), 1523–1537.

Fan, X., & Arcak, M. (2003). Observer design for systems with multi-variable monotone nonlinearities. *Systems and Control Letters*, *50*, 319–330.

Glumineau, A., Moog, C. H., & Plestan, F. (1996). New algebro-geometric conditions for the linearization by input–output injection. *IEEE Transactions on Automatic Control*, *41*(4), 598–603.

Haddad, W. M., & Bernstein, D. S. (1994). Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle, and Popov theorems and their application to robust stability. Part ii: Discrete-time theory. *International Journal of Robust and Nonlinear Control*, *4*, 249–265.

Ibrir, S. (2003). On-line exact differentiation and notion of asymptotic algebraic observers. *IEEE Transactions on Automatic Control*, *48*(11), 2055–2060.

Ibrir, S., Xie, W. F., & Su, C.-Y. (2005). Observer-based control of discrete-time Lipschitzian nonlinear systems: Application to one-link flexible joint robot. *International Journal of Control*, *78*(6), 385–395.

Kapila, V., & Haddad, W. M. (1996). A multi-variable extension of the Tsypkin criterion using a Lyapunov-function approach. *IEEE Transactions on Automatic Control*, *41*(1), 149–152.

Kazantzis, N., & Kravaris, C. (1998). Nonlinear observer design using Lyapunov's auxiliary theorem. *Systems and Control Letters*, *34*(5), 241–247.

Krener, A. J., & Respondek, W. (1985). Nonlinear observers with linearizable error dynamics. *SIAM Journal on Control and Optimization*, *23*(2), 197–216.

Lee, W., & Nam, K. (1991). Observer design for autonomous discrete-time nonlinear systems. *Systems and Control Letters*, *17*, 49–58.

Moraal, P. E., & Grizzle, J. W. (1995). Observer design for nonlinear systems with discrete-time measurements. *IEEE Transactions on Automatic Control*, *40*(3), 395–404.

Oliveira, M. C., Bernussou, J., & Geromel, J. C. (1999). A new discrete-time robust stability condition. *Systems and Control Letters*, *37*(4), 261–265.

Raghavan, S., & Hedrick, J. K. (1994). Observer design for a class of nonlinear systems. *International Journal of Control*, *59*(2), 515–528.

Rajamani, R. (1998). Observers for Lipschitz nonlinear systems. *IEEE Transactions on Automatic Control*, *43*(3), 397–400.

Rajamani, R., & Cho, Y. M. (1998). Existence and design of observers for nonlinear systems: Relation to distance to unobservability. *International Journal of Control*, *69*(5), 717–731.

Reif, K., Günther, S., Yaz, E., & Unbehauen, R. (1999). Stochastic stability of the discrete-time extended Kalman filter. *IEEE Transactions on Automatic Control*, *44*(4), 714–728.

Ren, J., & Guo, L. (2005). An impossibility theorem on sampled-data feedback of uncertain nonlinear systems. In *Proceedings of the international conference on control and automation* (pp. 53–58), Hungary, June.

Richter, H., & Misawa, E. (2003). Stability of discrete-time systems with quantized input and state measurements. *IEEE Transactions on Automatic Control*, *48*(3), 1453–1458.

Shim, H., Son, Y. I., & Seo, J. H. (2000). Semi-global observer for multi-output nonlinear systems. *Systems and Control Letters*, *42*(3), 233–244.

Slotine, J. J. E., Hedrick, J. K., & Misawa, E. A. (1987). On sliding observers for nonlinear systems. *Journal of Dynamic Systems, Measurement, and Control*, *109*, 245–252.

Thau, F. E. (1973). Observing the state of nonlinear dynamic systems. *International Journal of Control*, *17*, 471–479.

Tornambè, A. (1992). High-gain observers for nonlinear systems. *International Journal of Systems Science*, *23*, 1475–1489.

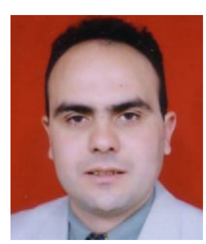
Wu, S. H. (1967). A circle criteria for a class of discrete systems. *IEEE Transactions on Automatic Control*, *12*, 114–115.

Xia, X. H., & Gao, W.-B. (1989). Nonlinear observer design by observer error linearization. *SIAM Journal on Control and Optimization*, *27*(1), 199–216.

Xie, L.-L., & Guo, L. (2000). How much uncertainty can be dealt with by feedback. *IEEE Transactions on Automatic Control*, *45*(12), 2203–2217.

Xue, F., & Guo, L. (2002). On limitations of the sampled-data feedback for nonparametric dynamical systems. *Journal of Systems Science and Complexity*, *15*(3), 225–250.

Yaz, E., & Azemi, A. (1993). Sliding mode observers for nonlinear models with unbounded noise and measurement uncertainties. *Dynamics and Control*, *3*(3), 217–235.



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