OBSERVATION OF HAMILTONIAN MECHANICAL SYSTEMS.

Salim IBRIR

Laboratoire des Signaux et Systèmes, Plateau de Moulon 91192
Gif sur Yvette Cedex, France.
Tel : +33 1 69851756, Fax: +33 1 69413060
E-mail : ibrir@lss.supelec.fr

ABSTRACT

The design of observers plays a key role in control of nonlinear systems. In this paper we shall consider the problem of nonlinear observers for Hamiltonian mechanical systems whose output is the collection of the generalized coordinates. It will be shown that these systems are equivalent to linear time-varying systems after injection of the first derivative of the output. The analysis of the existence of an observer is then based upon the theory of time-varying systems. Two differentiator observers are developed.

Keywords : Nonlinear observers, Time-varying systems, Sliding mode, Kalman observers, System theory.

1. INTRODUCTION

The design of nonlinear observers is one of the crucial points in control theory and has received a great deal of interest since the appearance of Luenberger's observer [15]. Significant contributions to the theory of nonlinear observers were given by Thau [21], Krener and Isodori [13], Krener and Respondek [14], Bastin and Gevers [1], Xia and Gao [27], Tornambè ([23], [22]), Marino [16], Tsinias [24], Gauthier et al [6], Krener [12]. Unfortunately, the developed observers depend on a set of geometric conditions under which the dynamics of the observation error is linear. The linearization by input-output injection has appeared as a powerful tool to linearize the dynamics of nonlinear observers, but no solution has been proposed to solve the problem of estimation of the derivatives of the inputs and the outputs, see ([7], [17], [11], [3], [8]). In [4], the authors have developed a new method of observer design which consists in estimating the unmeasured states from a static diffeomorphism using numerical differentiation techniques. Other results for this type of observation are discussed in the literature ([9], [18]).

Mechanical systems have emerged as the most promising application area for the theory of nonlinear observers, because of the availability of dynamical models and special properties they frequently possess that simplify the analysis of the existence of observers. In this article we focus our interest upon the observation problem of Euler-Lagrange systems, in which the generalized coordinates are given by $q = (\, q_1 \quad q_2 \quad \cdots \quad q_n \,)^T$ and the equations of motion are

$$\frac{d}{dt}\left\{\frac{\partial L(q,\,\dot q)}{\partial \dot q}\right\} - \frac{\partial L(q,\,\dot q)}{\partial q} = -\frac{\partial F(\dot q)}{\partial \dot q} + Q\,u, \qquad (1)$$

where L is the Lagrangian, defined by $L(q,\,\dot q) = T(q,\,\dot q) - V(q)$, $T(q,\,\dot q)$ is the kinetic energy, $V(q)$ stands for the potential energy, and the dot denotes differentiation with respect to time t. We note $F(\dot q)$ the external forces due to dissipation, u is the control inputs, and Q is a real matrix. Equations (1) are written in *Newton second law* form as follows :

$$M(q)\,\ddot q + C(q,\,\dot q)\,\dot q + \frac{\partial V(q)}{\partial q} + \frac{\partial F(\dot q)}{\partial \dot q} = Q\,u, \qquad (2)$$

where $M(q)$ is a symmetric, positive definite matrix called the inertia matrix. The term $C(q,\,\dot q)$ expresses the workless forces (*Coriolis and centrifugal forces*) such that $\dot M(q) = C(q,\,\dot q) + C^T(q,\,\dot q)$. $G(q)$ is the vector of forces derived from the variation of the potential energy $V(q)$. Notice that if the vector $y = q$ is fully measured, and $\frac{\partial F(\dot q)}{\partial \dot q} = C_f\,\dot q$, then we can write the dynamic equations (2) as follows :

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} q \\ \dot q\end{pmatrix} &= \begin{pmatrix} 0 & I_{n\times n} \\ 0 & -M^{-1}(q)(C(q,\,\dot q)+C_f)\end{pmatrix}\begin{pmatrix} q \\ \dot q\end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ \tau(q,u)\end{pmatrix}, \qquad (3)\\ y &= q, \qquad (4)\end{aligned}$$

where $\tau(q,\,u) = M^{-1}(q)(Q\,u - G(q))$. Finally, if we note the state vector $x = (\, q \quad \dot q\,)^t \in \mathbb{R}^{2n\times 2n}$, then system (2) takes the form :

$$\begin{aligned}\dot x &= \mathcal{F}(y,\,\dot y)\,x + \mathcal{G}(y,\,u). \qquad (5)\\ y &= C\,x. \qquad (6)\end{aligned}$$

The resulting state-affine system involves estimation of the first derivatives of the output, for this reason, section 2 is devoted to construction of two asymptotic differentiator observers that can estimate the higher derivatives of any scalar measured variable.

2. TWO ASYMPTOTIC DIFFERENTIATOR OBSERVERS

In this subsection, we shall present two observers of order n which estimate the higher derivatives of any continuous measured output. The first observer is a smooth observer with a dynamic gain, while the second is a discontinuous observer with a static gain. We shall show that the differentiator given by Esfandari and Khalil in [5] is a special case of the first developed observer.

2.1. A differentiator observer with a dynamic gain

Theorem 2.1 *Let $y : \mathbb{R}_{>0} \to \mathbb{R}$ be a scalar measurable function. For γ large enough, the differentiator observer of order n*

$$\dot{\hat{x}} = A_{n\times n}\,\hat{x} + L^{-1}(t)\,C^T_{1\times n}\,(y - C_{1\times n}\,\hat{x}), \quad (7)$$

approximate the derivatives of the variable y up to the order $n-1$ when $t \to \infty$, where the matrix $L : \mathbb{R}_{>0} \to \mathbb{R}^{n\times n}$ is the solution of the dynamic equation

$$\dot{L} = -\gamma L - A^T_{n\times n}\,L - L\,A_{n\times n} + C^T_{1\times n}\,C_{1\times n}, \quad (8)$$

such that $A_{i,j} = \delta_{i,j-1}$ and ($C_i = 0$ if $i \neq 1$, $C_1 = 1$).

Proof. It is sufficient to compute the transfer functions of the observer states with respect to the output y when the gain matrix $L(t)$ converges to a stationary matrix L_∞. It is easy to prove that when $t \to \infty$, L converges to a constant matrix L_∞ of the following type $(L_\infty)_{i,j} := \frac{(-1)^{i+j}\,|\alpha_{i,j}|}{\gamma^{i+j-1}}$, where $\alpha \in \mathbb{R}^{n\times n}$ is the solution of

$$-\gamma\,\alpha - A^T_{n\times n}\,\alpha - \alpha\,A_{n\times n} + C^T_{1\times n}\,C_{1\times n} = 0. \quad (9)$$

Taking the Laplace transform of equation (21) and replacing L by L_∞, the transfer functions $\frac{\hat{X}(s)}{Y(s)}$ become

$$\frac{\hat{X}(s)}{Y(s)} = \left(sI - A + L_\infty{}^{-1}\,C^T\,C\right)^{-1} L_\infty{}^{-1}C^T \quad (10)$$

$$= \frac{\left(N_1(s) \quad N_2(s) \quad \cdots \quad N_n(s)\right)^T}{(s+\gamma)^n} \quad (11)$$

where each $N_j(s)$ is given by

$$N_j(s) = \sum_{k=j}^{n} C_n^k\,\gamma^k\,s^{n-k+j-1} \quad (12)$$

Then we obtain

$$\frac{\hat{X}_j(s)}{Y(s)} == \frac{\sum_{k=j}^{n} C_n^k\,\gamma^k\,s^{n-k+j-1}}{\sum_{k=0}^{n} C_n^k\,\gamma^k\,s^{n-k}}. \quad (13)$$

It is obvious that

$$\lim_{\gamma\to\infty} \frac{\sum_{k=j}^{n} C_n^k\,\gamma^k\,s^{n-k+j-1}}{\sum_{k=0}^{n} C_n^k\,\gamma^k\,s^{n-k}} = s^{j-1} \quad (14)$$

which translates that the observer state $\hat{X}_j$ approximates the $(j-1)$-th derivative of the output y. ■

Remark 2.1 *We have seen that the observer gain converges to the gain vector $(\,C_n^1\gamma \quad C_n^2\gamma^2 \quad \cdots \quad \gamma^n\,)^T$ which is the same as the one we would obtain with the hign gain observer (see [5])*

$$\begin{aligned}
\dot{x}_1 &= x_2 + \frac{\alpha_1}{\epsilon}(y - x_1),\\
&\vdots\\
\dot{x}_{n-1} &= x_{n-1} + \frac{\alpha_{n-1}}{\epsilon^{n-1}}(y - x_1),\\
\dot{x}_{n-1} &= \frac{\alpha_n}{\epsilon^n}(y - x_1).
\end{aligned}$$

where $\gamma = \frac{1}{\epsilon}$ and ($\alpha_k = C_n^k$, $k = 1, \cdots, n$).

2.2. A sliding mode differentiator observer

Theorem 2.2 *For γ large enough the states of the following sliding observer*

$$(\Sigma)_s \begin{cases} \dot{\hat{x}}_1 = -\beta_1\,(\hat{x}_1 - y) + x_2 - k_1\,sign(\hat{x}_1 - y),\\ \vdots\\ \dot{\hat{x}}_{n-1} = -\beta_{n-1}\,(\hat{x}_1 - y) - k_{n-1}sign(\hat{x}_1 - y),\\ \dot{\hat{x}}_n = -\beta_n\,(\hat{x}_1 - y) - k_n sign(\hat{x}_1 - y), \end{cases}$$

approximate the successive derivatives of the scalar function y. The constants (β_ℓ, $\ell = 1, \cdots, n$) are supposed to be positive and the gain coefficients (k_ℓ, $\ell = 1, \cdots, n$) are determined by the formula

$$\begin{cases} k_1 := \gamma & \forall n,\\ k_\ell := C_{n-1}^{\ell-1}\gamma^\ell, & \ell = 2, \cdots, n;\ n \geq 2, \end{cases}$$

Proof. The proof is quite similar to the proof of the first theorem.

3. THE PRACTICAL OBSERVER : UPDATING THE DERIVATIVES

Since the unmeasured states are the first derivatives of the generalized coordinates, the observer discussed in this section shall update the derivatives computed by the asymptotic differentiator observers by constructing another asymptotic observer whose gain is the solution of Ricatti equation. See ([2], [26], [19], [28], [25], [10]) for more details. We summarize the main result of this section in the following statement

Theorem 3.1 *Consider the system (5), (6). Let* $\xi_\ell := \begin{pmatrix} y_\ell \\ \dot{y}_\ell \end{pmatrix}$ *be the ℓ-th state vector of the system family*

$$\begin{cases} \dot{\widehat{\xi}}_\ell = A_{2\times 2}\widehat{\xi}_\ell + L_{2\times 2}^{-1}C_\xi^T(y_\ell - C_\xi\widehat{\xi}_\ell), & \ell = 1, \cdots, n, \\ \dot{L}_{2\times 2}(t) = -\gamma\, L_{2\times 2}(t) - A_{2\times 2}^T L_{2\times 2}(t) \\ \qquad -L_{2\times 2}(t)A_{2\times 2} + C_\xi{}^T\, C_\xi, \end{cases}$$

such that $C_\xi = (1 \quad 0)$, $A_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. *If* $R(t) : \mathbb{R}_{>0} \to \mathbb{R}^{2n\times 2n}$ *is a symmetric, uniformely positive definite, and bounded matrix, then the dynamic system*

$$\begin{cases} \dot{\widehat{x}} = \mathcal{F}(y, \widehat{\xi})\,\widehat{x} + P(t)C^T R(t)(y - C\widehat{x}) + \mathcal{G}(y, u), \\ \dot{P}(t) = \mathcal{F}(y, \widehat{\xi})P + P\mathcal{F}^T(y, \widehat{\xi}) - P(t)C^T R(t)CP(t), \end{cases} \tag{15}$$

is an observer of the system (5), (6) for $\gamma \to \infty$ *and for any* $P(0) > 0$.

4. NUMERICAL SIMULATIONS

Consider the Euler-Lagrange mechanical system depicted in Fig. 1. This system has two degrees of freedom $q = (x, \theta)$ where x is the displacement of the bowl and θ is the angle of the cylinder that rolls inside the bowl without sliding. Applying the Lagrange formulation (1), then equations of motion are given by

$$\begin{aligned} \dot{q}_1 &= \dot{y}_1, \\ \dot{q}_2 &= \dot{y}_2, \\ \ddot{q}_1 &= \frac{-J_{eq}(ky_1+\lambda\dot{y}_2^2\cos y_2)+\lambda^2 g\sin^2 y_2}{M_{eq}J_{eq}-(\lambda\sin y_2)^2} + \frac{J_{eq}}{M_{eq}J_{eq}-(\lambda\sin y_2)^2}\,u, \\ \ddot{q}_2 &= \frac{\lambda\sin y_2(ky_1+\lambda\dot{y}_2^2\cos y_2)-M_{eq}\lambda g\sin y_2}{M_{eq}J_{eq}-(\lambda\sin y_2)^2} - \frac{\lambda\sin y_2}{M_{eq}J_{eq}-(\lambda\sin y_2)^2}\,u, \end{aligned}$$

In order to build an observer for (Σ), let us note the state vector $x = (y_1, y_2, \dot{y}_1, \dot{y}_2)^T$. Then (Σ) is put in form (5) such that

$$\mathcal{F}(y, \dot{y}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{-J_{eq}\lambda\dot{y}_2\cos y_2}{M_{eq}J_{eq}-(\lambda\sin y_2)^2} \\ 0 & 0 & 0 & \frac{\lambda^2\sin y_2\cos y_2\dot{y}_2}{M_{eq}J_{eq}-(\lambda\sin y_2)^2} \end{pmatrix}, \tag{16}$$

$$\mathcal{G}(y, u) = \begin{pmatrix} 0 \\ 0 \\ \frac{J_{eq}\,u - J_{eq}\,k\,y_1+\lambda^2 g\sin^2 y_2}{M_{eq}J_{eq}-(\lambda\sin y_2)^2} \\ \frac{-\lambda\,u\sin y_2+\lambda\,k\,y_1\sin y_2-M_{eq}\lambda g\sin y_2}{M_{eq}J_{eq}-(\lambda\sin y_2)^2} \end{pmatrix}. \tag{17}$$

In the simulations represented below, we have taken $x_0 = (0 \quad 0.4 \quad 0 \quad 0)$, $\widehat{x}_0 = (15 \quad 20 \quad -10 \quad -2)^T$, $u(t) =$

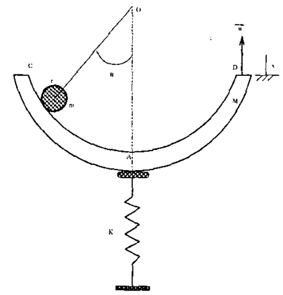

Figure 1: The mechanical system.

$2\sin(t)$. The matrix R is set to the constant identity matrix of dimension 2. In Fig. 2 we present the velocity of the bowl and its estimate. Fig. 3 shows the exact angular velocity of the cylinder and its corresponding observer state. All the observer states are ploted in dash lines.

5. CONCLUSIONS

In this paper we proposed two approaches of observer design for Euler-Lagrange mechanical systems where all the generalized coordinates are fully measured. The first approach consists in estimating the velocities without any information of the system dynamics but only upon the measured coordinates. Two differentiator observers are developed for this purpose. We pointed out that the differentiator observer of the dynamic gain converges to the high gain observer proposed in [5] which is based on singular perturbation theory. Furthermore, the developed differentiator observer offers a nice transient behavior because of the variation of the observer gain. The second approach of observation uses the states of the differentiator observer to build a simple Kalman observer for the mechanical system, considered as a linear time-varying system. It is shown that the existence of the observer is free from any geometric condition and it can be used for any Euler-Lagrange system.

6. REFERENCES

[1] G. Bastin and M. Gevers. "Stable adaptive observers for nonlinear time-varying systems". *IEEE Transactions on Automatic Control*, **33**(7):650–658, July 1988.

[2] R. S. Bucy. "Canonical forms for multivariable systems". *IEEE Transactions on Automatic Control, short paper*, pages 567–569, October 1968.

[3] D. Cheng, A. Isodori, W. Respondek, and T. J. Tarn. "Exact linearization of nonlinear systems with outputs". *Math. Systems Theory*, **21**:63–83, 1988.

[4] S. Diop, J. W. Grizzle, P. E. Morral, and A. G. Stefanoupoulou. "Interpolation and numerical differentiation for observer design". *In proceedings of the Americal Control Conference, Americain Control Consil*, pages 1329–1333, 1993. Evanston, IL.

[5] F. Esfandari and K. H. Khalil. "Output feedback stabilization of fully linearizable systems". *Int. J. Control*, **56**, 1992.

[6] J. P. Ghautier, H. Hammouri, and S. Othman. "A simple observer for nonlinear systems: Application to bioreactors". *IEEE trans. Automat. Control*, **37**(6):875–880, June 1992.

[7] A. Glumineau, C. H. Moog, and F. Plestan. "New algebro-geometric conditions for the linearization by input-output injection". *IEEE Transactions on Automatic Control*, **41**(4):598–603, April 1996.

[8] H. Hammouri and J. de Leon Morales. "Observer synthesis for state-affine systems". *In Proceedings of the 29-th Conference on Decision and Control, Honolulu, Hawaii*, December 1990.

[9] S. Ibrir. "A numerical algorithm for filtering and state observation". *In proceedings of IEEE, International Conference on Acoustics, Speech, and Signal processing (ICASSP), May 1998*, May 1998. Washinghton.

[10] M. Ikeda, H. Maeda, and S. Kodama. "Estimation and feedback in linear time-varying systems: A deterministic theory". *SIAM J. Control*, **13**(2):304–326, February 1975.

[11] M. Jankovic. "A new observer for a class of nonlinear systems". *Journal of Mathematical Systems, Estimation, and Control*, **3**(2):225–246, 1993.

[12] W. Kang and A. Krener. "Nonlinear observer design, a backstepping approach". *Technical note*, 1998.

[13] A. J. Krener and A. Isodori. "Linearization by output injection and nonlinear observers". *Systems and control letters*, **3**:47–52, 1983.

[14] A. J. Krener and W. Respondek. "Nonlinear observers with linearizable error dynamics". *SIAM J. Control and optimization*, **23**(2), 1985.

[15] D. J. Luenberger. "An introduction to observers". *IEEE trans. Automat. Control*, **AC-16**(6):596–602, December 1971.

[16] R. Marino. "Adaptive observers for single output nonlinear systems". *IEEE Transactions on Automatic Control*, **35**(9):1054–1058, September 1990.

[17] F. Plestan and A. Glumineau. "Linearization by generalized input-output injection". *Systems & Control Letters*, **31**:115–128, 1997.

[18] F. Plestan and J. Grizzle. "Synthesis of nonlinear observers via structural analysis and numerical differentiation". *Technical note*, 1998.

[19] K. Ramar and B. Ramaswami. "Transformation of time-variable multi-input systems to a canonical form". *IEEE Transactions on Automatic Control, technical notes and correspondence*, pages 371–374, December 1970.

[20] J. J. E. Slotine, J. K. Hedrick, and E. A. Misawa. "On sliding observers for nonlinear systems". *Journal of dynamic systems, Measurement, and Control*, **109**:245–252, 1987.

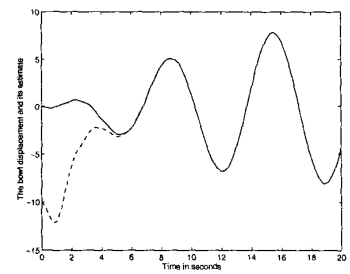

Figure 2: The exact velocity of the bowl and its estimate.

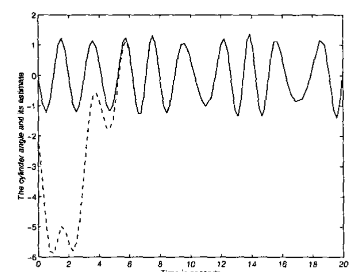

Figure 3: The exact angular velocity of the cylinder and its estimate.

[21] F. E. Thau. "Observing the state of nonlinear dynamic systems". *International Journal of Control*, **17**:471–479, 1973.

[22] A. Tornambè. "Asymptotic observers for nonlinear systems". *International J. Systems Sci.*, **23**(3):435–442, 1992.

[23] A. Tornambè. "High gain observers for nonlinear systems". *International Journal J. Systems Sci.*, **23** (9):1475–1489, 1992.

[24] J. Tsinias. "Further results on the observer design problem". *Systemes & Control Letters*, **14**:411–418, 1990.

[25] J. C. Willems and S. K. Mitter. "Controllability, observability, pole allocation and state reconstruction". *IEEE Transactions on Automatic Control*, **AC-16**(6):582–595, December 1971.

[26] W. A. Wolovich. "On the stabilization of controlloble systems". *IEEE Transactions on Automatic Control, short paper*, pages 569–572, October 1968.

[27] X. Xia and W. Gao. "On exponential observers for nonlinear systems". *Systems & Control Letters*, **11**(4):319–326, 1988.

[28] Y. O. Yüksel and J. Bongiorno. "Observers for linear multivariable systems with applications". *IEEE Transactions on Automatic Control*, **AC-16**(6):603–612, December 1971.