Circle-criterion observers for dynamical systems with positive and non-positive slope nonlinearities

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***Abstract*— In this paper, we give new sufficient linear matrix inequality conditions that guarantee the existence of globally converging observers for systems with both positive and non-positive slope nonlinearities. The proposed design method is basically founded on the concept of the circle criterion in continuous-time. Many examples are studied to show both the novelty and the efficacy of the obtained theoretical results.**

***Index Terms*— Nonlinear observers; Multiple-output-injection observers; Circle criterion; Linear Matrix Inequalities (LMIs).**

I. Introduction

Circle-criterion approach to nonlinear observer design becomes one of the popular methodologies that simplifies the design of converging observers for certain class of inherently nonlinear systems [1]. This is mainly due to the fact that the observer formulation is given in convex optimization setting allowing numerical tractability of the solutions. Due to the fascinating theoretical as well as the proven practical applicability of convex optimization procedures, the LMI-based designs have emerged as powerful tools that permit to solve complex observation issues and multi-objectives estimation design problems in a smart and convenient way [2]. Therefore, solutions of nonlinear observer design problems through convex optimization techniques are considered as practical solutions. The circle-criterion observers for nonlinear systems with slope-restricted nonlinearities have been proposed by Arcak and Kokotović [3]. More general results have been obtained for systems with multi-variable monotone nonlinearities, see [4], [5]. In [4] the authors proposed a linear matrix inequality condition that guarantees the existence of globally converging observers for systems described by the following dynamical equations

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + G\,\gamma\big(H\,x(t)\big) + g(u(t), y(t)),\\ y(t) &= C\,x(t), \end{aligned} \tag{1}$$

where A, C, G and H are real matrices of appropriate dimensions, $g(u(t), y(t))$ is an arbitrary real-valued vector that depends on the system inputs and outputs and $\gamma(\cdot)$ is a vector nonlinearity that verifies

$$\left(\frac{\partial\gamma(s)}{\partial s}\right)' + \left(\frac{\partial\gamma(s)}{\partial s}\right) \geq 0. \tag{2}$$

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For system (1), a nonlinear observer has been proposed in [4] as follows

$$\begin{aligned} \dot{\hat{x}}(t) &= A\,\hat{x}(t) + \underbrace{G\,\gamma\big(H\,\hat{x}(t) + K\big(C\hat{x}(t) - Cx(t)\big)\big)}_{\ell_1(\hat{x}(t), y(t))}\\ &\quad + g(u(t), y(t)) + \underbrace{L(C\hat{x}(t) - Cx(t))}_{\ell_2(y(t), \hat{y}(t))}, \\ \hat{y}(t) &= C\,\hat{x}(t), \end{aligned} \tag{3}$$

where $\ell_1(\cdot,\cdot)$ and $\ell_2(\cdot,\cdot)$ stand for the nonlinear and the linear output injection terms, respectively. In fact, condition (2) becomes quite strong if the system under consideration contains different kinds of nonlinearities. For example, nonlinearities with non-positive slopes as x^2, x^4 and $x_1 x_2$ cannot verify the growth condition (2). Therefore, the design proposed in [4] becomes no longer valid if condition (2) is not satisfied at any time. The main objective of this paper is to propose a novel observation procedure that deals with both positive and non-positive slope nonlinearities. The basic difference between the proposed algorithm and that proposed in [4] is the employment of different nonlinear output injection terms to feedback the observer. The conditions of the observer convergence are derived through the circle criterion and are expressed in terms of numerically tractable linear matrix inequalities. We show through several examples that the developed technique is quite useful, in the sense that, both positive and non-positive slope nonlinearities are tolerated. Consequently, the circle-criterion observer design is extended to more general systems involving broad-spectrum of nonlinearities.

Throughout this paper, we note by $\mathbb{R}$ the set of real numbers. The notation $A > 0$ (resp. $A < 0$) means that the matrix A is positive definite (resp. negative definite). A' is the matrix transpose of A. $\dot{x}(t)$ stands for the time-derivative of the vector $x(t)$ with respect to time.

II. Observer design

In this section we plan to extend the circle-criterion observer design to more general systems involving nonlinearities that may not have positive slopes. In our design, nonlinearities with non-positive growth will be treated in the same way as nonlinearities having positive gradients. Before giving the main result of this section, let us begin by exposing some motivating examples.

A. *Motivating examples*

As we have mentioned before, nonlinearity as x^2 has not always a positive slope. However, it could be expanded as a linear combination of nonlinear functions with positive slopes. One of the possible expansions is

$$\begin{aligned} x^2 &= \frac{1}{3}(x+1)^3 - \frac{1}{3}x^3 - x - \frac{1}{3} \\ &= -\frac{1}{3} + \sum_{i=1}^{3} \alpha_i\, f_i(x+\xi_i) \end{aligned} \tag{4}$$

where $\alpha_1 = \frac{1}{3}$, $\alpha_2 = -\frac{1}{3}$, $\alpha_3 = -1$, $\xi_1 = 1$, $\xi_2 = \xi_3 = 0$, $f_1(x) = f_2(x) = x^3$ and $f_3(x) = x$. This is not the only way to rewrite the x^2 nonlinearity, one can also rewrite any term of the form $\beta\, x^2$, $\beta > 0$ as follows

$$\begin{aligned} \beta\, x^2 &= (\beta\, x^2 + \alpha\, x^3 + \delta\, x) - \alpha\, x^3 - \delta\, x, \\ &= f_1(x) - \alpha\, f_2(x) - \delta\, f_3(x), \end{aligned} \tag{5}$$

where $f_1(x) = \beta\, x^2 + \alpha\, x^3 + \delta\, x$, $f_2(x) = x^3$ and $f_3(x) = x$. If we chose α and δ such that $\alpha > 0$, and $\alpha\, \delta > \frac{1}{3}\beta^2$ then, $f_1(x)$, $f_2(x)$ and $f_3(x)$ have all positive slopes. Nonlinearities of the form $x_1 x_2$ are generally encountered in practical systems and can also be expanded as

$$\begin{aligned} x_1 x_2 &= (x_1 + \frac{1}{2}x_2)^2 - x_1^2 - \frac{1}{4}x_2^2 \\ &= \frac{1}{12} - \frac{1}{4}x_2 + \frac{1}{3}(x_1 + \frac{1}{2}x_2 + 1)^3 \\ &\quad - \frac{1}{3}(x_1 + \frac{1}{2}x_2)^3 - \frac{1}{3}(x_1+1)^3 + \frac{1}{3}x_1^3 \\ &\quad - \frac{1}{12}(x_2+1)^3 + \frac{1}{12}x_2^3 \\ &= \frac{1}{12} + \sum_{i=1}^{7} \beta_i\, f_i(H_i[x_1\ x_2]' + \xi_i) \end{aligned} \tag{6}$$

where $\beta_1 = -\frac{1}{4}$, $\beta_2 = \frac{1}{3}$, $\beta_3 = -\frac{1}{3}$, $\beta_4 = -\frac{1}{3}$, $\beta_5 = \frac{1}{3}$, $\beta_6 = -\frac{1}{12}$, $\beta_7 = \frac{1}{12}$, $f_1(x) = x$, $(f_i(x))_{2\le i\le 7} = x^3$, $H_1 = H_6 = H_7 = [0\ 1]$, $H_2 = H_3 = [1\ \frac{1}{2}]$, $H_4 = H_5 = [1\ 0]$, $\xi_1 = \xi_3 = \xi_5 = \xi_7 = 0$ and $\xi_2 = \xi_4 = \xi_6 = 1$.

From the previous illustrations, a single nonlinearity can be seen as an algebraic combination of positive-slope nonlinearities. In this case, the use of a single nonlinear output injection term turns out to be impossible. By the use of the concept of the circle criterion, one can investigate the possibility of building a nonlinear observer with different output injection terms and hence, nonlinearities which may not have a positive slope can be treated in the same way as positive slope nonlinearities.

B. Main result

The first step towards the observer design is to decompose the system nonlinearity into a sum of positive-slope nonlinearities and then exploit each nonlinearity in the design of the nonlinear observer. To this end, assume that the nonlinear system admits the following representation

$$\begin{aligned} \dot{x}(t) &= A\, x(t) \\ &\quad + \sum_{i=1}^{\mu} G_i \pi_i(y(t))\, f_i\Big(H_i\, x(t) + \varphi_i(u(t), y(t)) + \xi_i\Big) \\ &\quad + g(u(t), y(t)) + W, \\ y(t) &= C\, x(t), \end{aligned} \tag{7}$$

where $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, $(G_i)_{1\le i\le \mu} \in \mathbb{R}^{n\times 1}$, $(H_i)_{1\le i\le \mu} \in \mathbb{R}^{1\times n}$ and $W \in \mathbb{R}^{n\times 1}$ are constant matrices and $(\xi_i)_{1\le i\le \mu} \in \mathbb{R}$ are known reals constants. We assume that the pair (A, C) is observable. $(\pi_i(y(t)))_{1\le i\le \mu}$ are positive scalar functions that depend on the system output. The smooth nonlinearities $(f_i(s(t)))_{1\le i\le \mu}$ are assumed to verify the following growth conditions

$$\frac{\mathrm{d} f_i(s(t))}{\mathrm{d} s(t)} \ge 0,\ 1 \le i \le \mu,\ \forall s(t) \in \mathbb{R}, \tag{8}$$

while $g(u(t), y(t)) \in \mathbb{R}^{n\times 1}$ and $(\varphi_i(u(t), y(t)))_{1\le i\le \mu}$ are arbitrary nonlinearities that may depend upon the system input $u(t) \in \mathbb{R}^m$ and the output $y(t) \in \mathbb{R}^p$. The objective is to design a nonlinear converging observer of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A\, \hat{x}(t) + g(u(t), y(t)) \\ &\quad + \sum_{i=1}^{\mu} \ell_i(\hat{x}(t), y(t), u(t)) + \ell_{\mu+1}(y(t), \hat{y}(t)) \\ &= A\, \hat{x}(t) + g(u(t), y(t)) \\ &\quad + \sum_{i=1}^{\mu} G_i \pi_i(y(t))\, f_i\Big(H_i\, \hat{x}(t) + \varphi_i(u(t), y(t)) \\ &\quad + K_i(\hat{y}(t) - y(t)) + \xi_i\Big) + W + L(\hat{y}(t) - y(t)), \\ \hat{y}(t) &= C\, \hat{x}(t), \end{aligned} \tag{9}$$

where $(K_i)_{1\le i\le \mu}$ and L are the nonlinear and the linear observer gains, respectively. $\ell_i(\hat{x}(t), y(t), u(t)) = G_i \pi_i(y(t))\, f_i\Big(H_i\, \hat{x}(t) + \varphi_i(u(t), y(t) + K_i(\hat{y}(t) - y(t)) + \xi_i\Big) + W$; $1 \le i \le \mu$, are the multiple nonlinear output injection terms and $\ell_{\mu+1}(y(t), \hat{y}(t)) = L(\hat{y}(t) - y(t))$ is the usual linear output injection term that guarantees the stability of the linear part of the observation-error dynamics. The design of the globally converging observer is given by the following statement.

Theorem 1: Consider the nonlinear system (7). If there exist a symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$, a matrix $Y \in \mathbb{R}^{n\times p}$ and a set of row vectors $(K_i)_{1\le i\le \mu} \in \mathbb{R}^{1\times p}$ such that the following linear matrix inequalities hold

$$\begin{aligned} A'P + PA + YC + C'Y' < 0, \\ G_i'P = -H_i - K_i C,\ 1 \le i \le \mu, \end{aligned} \tag{10}$$

then, the states of the following observer

$$\begin{aligned}\dot{\hat{x}}(t) &= A\,\hat{x}(t)\\&+\sum_{i=1}^{\mu} G_i\pi_i(y(t))\, f_i\Big(H_i\,\hat{x}(t)+\varphi_i(u(t),y(t))\\&+K_i(\hat{y}(t)-y(t))+\xi_i\Big)+g(u(t),y(t))\\&+W+P^{-1}Y(\hat{y}(t)-y(t)),\\\hat{y}(t) &= C\,\hat{x}(t),\end{aligned} \tag{11}$$

converge asymptotically to those of system (7).

Proof: Let $e(t)=x(t)-\hat{x}(t)$. This implies that

$$\begin{aligned}&\dot{e}(t)=(A+P^{-1}YC)e(t)\\&+\sum_{i=1}^{\mu} G_i\,\pi_i(y(t))\, f_i\Big(H_i\,x(t)+\varphi_i(u(t),y(t))+\xi_i\Big)\\&-\sum_{i=1}^{\mu} G_i\,\pi_i(y(t))\, f_i\Big(H_i\,\hat{x}(t)+K_i(\hat{y}(t)-y(t))\\&+\varphi_i(u(t),y(t))+\xi_i\Big).\end{aligned} \tag{12}$$

Let us put

$$\begin{aligned}v_i(t)&=H_i\,x(t)+\varphi_i(u(t),y(t))+\xi_i,\ 1\le i\le\mu,\\w_i(t)&=H_i\,\hat{x}(t)+\varphi_i(u(t),y(t))+K_i(\hat{y}(t)-y(t))+\xi_i,\\&\quad 1\le i\le\mu.\end{aligned} \tag{13}$$

Then, by the mean value Theorem, the observer error can be rewritten as follows

$$\begin{aligned}&\dot{e}(t)=(A+P^{-1}YC)e(t)+\int_0^1\sum_{i=1}^{\mu} G_i\,\pi_i(y(t))\times\\&\left.\frac{\partial f_i(s(t))}{\partial s(t)}\right|_{s(t)=s_i^\star(t)}(H_i+K_iC)e(t)\,\mathrm{d}\lambda,\end{aligned} \tag{14}$$

where $s_i^\star(t)=v_i(t)-\lambda(v_i(t)-w_i(t))$. The last dynamics of the observation error (14) can be rewritten as follows

$$\begin{aligned}\dot{e}(t)&=\big(A+P^{-1}YC\big)e(t)+\sum_{i=1}^{\mu} G_i\phi_i(t,z_i),\\z_i&=\big(H_i+K_iC\big)e(t),\ 1\le i\le\mu,\end{aligned} \tag{15}$$

where $\phi_i(t,z_i) = \int_0^1 \pi_i(y(t))\left.\frac{\partial f_i(s(t))}{\partial s(t)}\right|_{s(t)=s_i^\star(t)}(H_i + K_iC)e(t)\,\mathrm{d}\lambda$. As a result, the observation error dynamics can be seen as the interconnection of a linear dynamics and a sum of time-varying nonlinearities verifying the growth conditions $z_i\phi_i(t,z_i)\ge 0$; $\forall i$. Let us associate the following Lyapunov function: $V(e(t))=e'(t)Pe(t)$ to the dynamics (14). Then, we obtain

$$\begin{aligned}&\dot{V}(e(t))=e'(t)(A'P+PA+YC+C'Y')e(t)\\&+2\int_0^1\sum_{i=1}^{\mu} e'(t)PG_i\,\pi_i(y(t))\times\\&\left.\frac{\partial f_i(s(t))}{\partial s(t)}\right|_{s(t)=s_i^\star(t)}(H_i+K_iC)e(t)\,\mathrm{d}\lambda.\end{aligned} \tag{16}$$

If the conditions

$$PG_i=-H_i'-C'K_i',\ 1\le i\le\mu, \tag{17}$$

hold then, the integral term in (16) becomes negative or equal to zero since the slope of all the nonlinearities are positive or equal to zero. Consequently, the derivative of the Lyapunov function takes the form

$$\begin{aligned}&\dot{V}(e(t))=e'(t)(A'P+PA+YC+C'Y')e(t)\\&-2\int_0^1\sum_{i=1}^{\mu} e'(t)PG_i\,\pi_i(y(t))\times\\&\qquad\left.\frac{\partial f_i(s(t))}{\partial s(t)}\right|_{s(t)=s_i^\star(t)}G_i'P\,e(t)\,\mathrm{d}\lambda.\end{aligned} \tag{18}$$

In order to ensure the negativity of $\dot{V}(e(t))$, the condition

$$A'P+PA+YC+C'Y'<0 \tag{19}$$

must hold for any observable pair (A,C). This ends the proof.

III. Illustrative examples

In this section, we present numerous examples that explain in detail the developed theory.

Example 1: Consider the nonlinear system

$$\begin{aligned}\dot{x}_1&=x_2,\\\dot{x}_2&=-x_2^3+x_2^2-x_1+u,\\\dot{x}_3&=x_2-x_3-x_2^3,\\y_1&=x_1+x_2-x_3,\\y_2&=x_1+x_2+x_3.\end{aligned} \tag{20}$$

The nonlinearity x_2^2 has not always a positive slope, therefore, we rewrite this nonlinearity as follows

$$x_2^2=\frac{1}{3}(x_2+1)^3-\frac{1}{3}x_2^3-x_2-\frac{1}{3}. \tag{21}$$

This permits us to rewrite the dynamics of (20) as follows

$$\begin{aligned}\dot{x}&=\underbrace{\begin{bmatrix}0&1&0\\-1&-1&0\\0&1&-1\end{bmatrix}}_{A}x+\underbrace{\begin{bmatrix}0\\-\frac{4}{3}\\-1\end{bmatrix}}_{G_1}\underbrace{x_2^3}_{f_1(x_2)}\\&+\underbrace{\begin{bmatrix}0\\\frac{1}{3}\\0\end{bmatrix}}_{G_2}\underbrace{(x_2+1)^3}_{f_2(x_2+1)}+\underbrace{\begin{bmatrix}0\\-\frac{1}{3}\\0\end{bmatrix}}_{W}+\underbrace{\begin{bmatrix}0\\u\\0\end{bmatrix}}_{g(u)},\\y&=\underbrace{\begin{bmatrix}1&1&-1\\1&1&1\end{bmatrix}}_{C}x.\end{aligned} \tag{22}$$

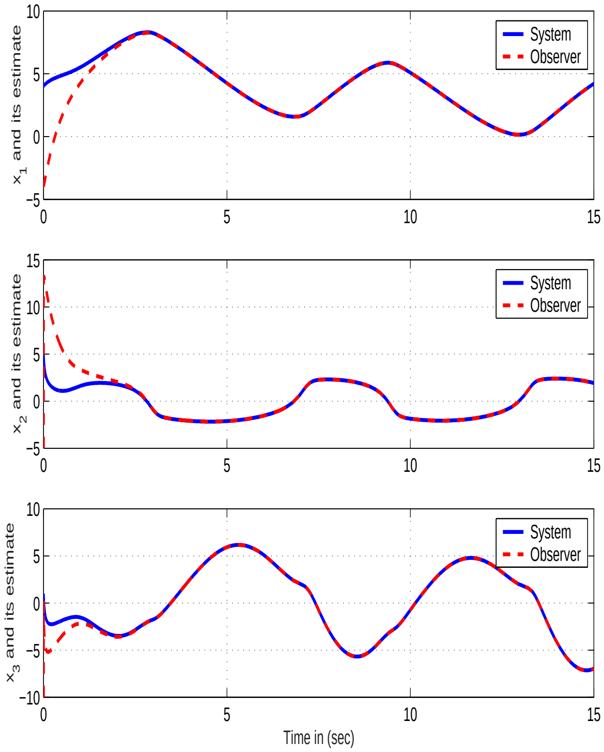

Fig. 1. The state vector x and its estimate $\hat{x}$

In this example $f_1(s) = f_2(s) = s^3$, $H_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = H_2$ and $\xi_1 = 0$, $\xi_2 = 1$. By solving the LMIs of Theorem 1, we get

$$\begin{aligned} P &= \begin{bmatrix} 21.2605 & 8.2912 & -7.4402 \\ 8.2912 & 5.2912 & -2.4402 \\ -7.4402 & -2.4402 & 8.4807 \end{bmatrix}, \\ Y &= \begin{bmatrix} 4.5791 & 0.2601 \\ -1.8261 & -2.1857 \\ -5.5613 & 0 \end{bmatrix}, \\ K_1 &= \begin{bmatrix} -0.8062 & 4.4209 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} -1.7886 - 0.9752 \end{bmatrix}. \end{aligned}$$

The nonlinear observer is readily constructed as

$$\begin{aligned} \dot{\hat{x}} = &\begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ -\frac{4}{3} \\ -1 \end{bmatrix} (\hat{x}_2 + K_1(\hat{y} - y))^3 \\ &+ \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \end{bmatrix} (\hat{x}_2 + K_2(\hat{y} - y) + 1)^3 + \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u \\ 0 \end{bmatrix} \\ &+ P^{-1} Y(\hat{y} - y), \; \hat{y} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \hat{x}. \end{aligned} \tag{23}$$

The behaviors of the system and the observer states for a periodic input $u(t) = 10\sin(t)$ are represented in Fig. 1.

Example 2: Consider the nonlinear plant described by the following dynamical equations

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2 x_2 + x_2^2 x_1, \\ \dot{x}_2 &= -x_2^3 - x_1 + u, \\ y &= x_1. \end{aligned} \tag{24}$$

The objective is to design a globally convergent observer for system (24) for all initial conditions $x_0 \in \mathbb{R}^2$. Remark that the nonlinearity $x_1^2 x_2 + x_2^2 x_1$ can be rewritten as follows

$$x_1^2 x_2 + x_2^2 x_1 = \frac{1}{3}(x_1 + x_2)^3 - \frac{1}{3} y^3 - \frac{1}{3} x_2^3. \tag{25}$$

According to the above expansion of the nonlinearity $x_1^2 x_2 + x_2^2 x_1$, the dynamics of (24) becomes

$$\begin{aligned} \dot{x} = &\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} -\frac{1}{3} \\ -1 \end{bmatrix}}_{G_1} (\underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{H_1} x)^3 \\ &+ \underbrace{\begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}}_{G_2} (\underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{H_2} x)^3 + \underbrace{\begin{bmatrix} -\frac{1}{3} y^3 \\ u \end{bmatrix}}_{g(u,y)}. \end{aligned} \tag{26}$$

After solving the LMIs of Theorem 1, we get

$$\begin{aligned} &P = \begin{bmatrix} 8.8604 & -3 \\ -3 & 2 \end{bmatrix}, \; Y = \begin{bmatrix} -5.0244 \\ -6.8604 \end{bmatrix}, \\ &K_1 = -0.0465, \; K_2 = -3.9535. \end{aligned} \tag{27}$$

The globally converging observer is then deduced

$$\begin{aligned} \dot{\hat{x}} = &\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \hat{x} + \underbrace{\begin{bmatrix} -\frac{1}{3} \\ -1 \end{bmatrix}}_{G_1} (\underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{H_1} \hat{x} + K_1(\hat{x}_1 - x_1))^3 \\ &+ \underbrace{\begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}}_{G_2} (\underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{H_2} \hat{x} + K_2(\hat{x}_1 - x_1))^3 \\ &+ \underbrace{\begin{bmatrix} -\frac{1}{3} y^3 \\ u \end{bmatrix}}_{g(u,y)} + P^{-1} Y(\hat{x}_1 - x_1). \end{aligned} \tag{28}$$

By taking the initial conditions as $x_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}'$, $\hat{x}_0 = \begin{bmatrix} -1 & -3 \end{bmatrix}'$ and $u(t) = \sin(t)$, the behaviors of the second state and its estimate are represented in Fig. 2.

Example 3: Let us consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + \sin(x_1)\, x_2^3, \\ \dot{x}_2 &= -x_2^3 + x_1\, u, \\ y &= x_1. \end{aligned} \tag{29}$$

The nonlinearity $\sin(x_1)\, x_2^3$ has not always a positive slope. Therefore, let us rewrite the last dynamical equations as follows

$$\begin{aligned} \dot{x}_1 &= x_2 + \big(\sin(y) + 1\big)\, x_2^3 - x_2^3, \\ \dot{x}_2 &= -x_2^3 + y\, u, \\ y &= x_1. \end{aligned} \tag{30}$$

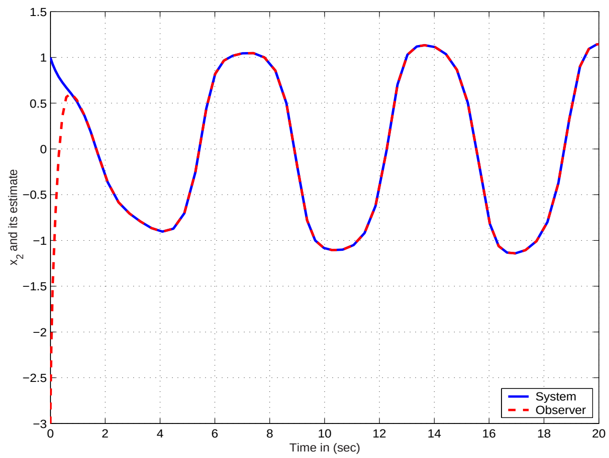

Fig. 2. x_2 and its estimate $\hat{x}_2$

According to this new representation, let us define

$$G_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ G_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},\ H_1 = H_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$\pi_1(y) = \sin(y) + 1, A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ g(u,y) = \begin{bmatrix} 0 \\ y\,u \end{bmatrix}. \tag{31}$$

By solving the LMIs of Theorem 1, we get

$$P = \begin{bmatrix} 2.7883 & -1 \\ -1 & 2 \end{bmatrix},\ Y_1 = \begin{bmatrix} -2.5445 \\ -1.7883 \end{bmatrix},$$
$$K_1 = -2.7883,\ K_2 = 1.7883. \tag{32}$$

The corresponding observer is then written as follows

$$\begin{aligned} \dot{\hat{x}} &= A\,\hat{x} + G_1\pi_1(y)(H_1\,\hat{x} + K_1(\hat{y} - y))^3 \\ &\quad + G_2(H_2\,\hat{x} + K_2(\hat{y} - y))^3 + g(u,\,y) + P^{-1}Y(\hat{y} - x_1), \\ \hat{y} &= \hat{x}_1. \end{aligned} \tag{33}$$

IV. LESS CONSERVATIVE CONDITION

When the number of nonlinearities increases, equality constraints as used in the statement of Theorem 1 become conservative conditions. In order to remove such equality constraints we will modify the conditions of convergence of the observation error to a non-strict LMIs. The design is summarized in the following statement.

Theorem 2: Consider the nonlinear system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) \\ &\quad + \sum_{i=1}^{\mu} G_i\pi_i(y(t))\,f_i\Big(H_i\,x(t) + \varphi_i(u(t),y(t)) + \xi_i\Big) \\ &\quad + g(u(t),y(t)) + W, \\ y(t) &= C\,x(t), \end{aligned} \tag{34}$$

satisfying the conditions of Theorem 1. If there exist a symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$, a matrix $Y \in \mathbb{R}^{n\times p}$, a set of reals $(\alpha_i)_{1\le i\le\mu} \in \mathbb{R}^{1\times p}$ and a set of row vectors $(K_i)_{1\le i\le\mu} \in \mathbb{R}^{1\times p}$ such that the following matrix inequalities hold

$$\begin{aligned} &A'P + PA + YC + C'Y' < 0, \\ &P\,G_i = \alpha_i\,G_i,\ 1 \le i \le \mu, \\ &P\,G_i\,H_i + G_i\,K_i\,C \le 0,\ 1 \le i \le \mu, \end{aligned} \tag{35}$$

then, the observation error $e(t) = \hat{x}(t) - x(t)$ is globally asymptotically stable where $\hat{x}(t)$ is the state vector of the following nonlinear observer

$$\begin{aligned} \dot{\hat{x}}(t) &= A\,\hat{x}(t) + \sum_{i=1}^{\mu} G_i\pi_i(y(t))\,f_i\Big(H_i\,\hat{x}(t) + \varphi_i(u(t),y(t)) \\ &\quad + \frac{K_i}{\alpha_i}(\hat{y}(t) - y(t)) + \xi_i\Big) + g(u(t),y(t)) \\ &\quad + W + P^{-1}Y(\hat{y}(t) - y(t)), \\ \hat{y}(t) &= C\,\hat{x}(t). \end{aligned} \tag{36}$$

Proof: From (16), if we replace K_i by $\dfrac{K_i}{\alpha_i}$, we can write that $\dot{V}(e(t) \le 0$ if $A'P + PA + YC + C'Y' < 0$ and $P\,G_i\,H_i + PG_i\,\frac{K_i}{\alpha_i}\,C \le 0$, $1 \le i \le \mu$. Using the constraints $P\,G_i = \alpha_i\,G_i$, $1 \le i \le \mu$ then, $P\,G_i\,H_i + G_i\,K_i\,C \le 0$, $1 \le i \le \mu$ are obtained. This ends the proof.

Remark that the constraints $P\,G_i = \alpha_i\,G_i$ are introduced to simplify the numerical solvability of the LMIs. But we can also remove the constraint $P\,G_i = \alpha_i\,G_i$ by solving a non-convex problem, that is

$$\begin{aligned} &A'P + PA + YC + C'Y' < 0, \\ &P\,G_i\,H_i + PG_i\,K_i\,C \le 0,\ 1 \le i \le \mu. \end{aligned} \tag{37}$$

The observer in this case is given by

$$\begin{aligned} \dot{\hat{x}}(t) &= A\,\hat{x}(t) \\ &\quad + \sum_{i=1}^{\mu} G_i\pi_i(y(t))\,f_i\Big(H_i\,\hat{x}(t) + \varphi_i(u(t),y(t)) \\ &\quad + K_i(\hat{y}(t) - y(t)) + \xi_i\Big) + g(u(t),y(t)) + W \\ &\quad + P^{-1}Y(\hat{y}(t) - y(t)), \\ \hat{y}(t) &= C\,\hat{x}(t). \end{aligned} \tag{38}$$

Example 4: Consider the nonlinear system described by the following dynamical equations

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1x_2, \\ \dot{x}_2 &= x_1 - x_2 + x_3 + u, \\ \dot{x}_3 &= -x_1 + x_2 + x_3 - x_1^3, \\ y_1 &= x_1, \\ y_2 &= x_2 + x_3. \end{aligned} \tag{39}$$

Remak that the only non-measured nonlinearity is x_1x_2. Using the following expansion

$$\begin{aligned} x_1x_2 &= \frac{1}{12} - \frac{1}{4}x_2 + \frac{1}{3}(x_1 + \frac{1}{2}x_2 + 1)^3 \\ &\quad - \frac{1}{3}(x_1 + \frac{1}{2}x_2)^3 - \frac{1}{3}(x_1 + 1)^3 + \frac{1}{3}x_1^3 \\ &\quad - \frac{1}{12}(x_2 + 1)^3 + \frac{1}{12}x_2^3, \end{aligned} \tag{40}$$

then, system (39) is rewritten as follows

$$\begin{aligned}\dot{x}_1 &= \frac{3}{4}x_2 + \frac{1}{3}(x_1 + \frac{1}{2}x_2 + 1)^3 \\ &\quad - \frac{1}{3}(x_1 + \frac{1}{2}x_2)^3 - \frac{1}{3}(y_1+1)^3 + \frac{1}{3}y_1^3 \\ &\quad - \frac{1}{12}(x_2+1)^3 + \frac{1}{12}x_2^3 + \frac{1}{12}, \\ \dot{x}_2 &= x_1 - x_2 + x_3 + u, \\ \dot{x}_3 &= -x_1 + x_2 + x_3 - x_1^3, \\ y_1 &= x_1, \\ y_2 &= x_2 + x_3. \end{aligned} \tag{41}$$

In matrix notation the last dynamical system is represented as

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & \frac{3}{4} & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} x + \underbrace{\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 0 \end{bmatrix}}_{G_1} \Big(\underbrace{\begin{bmatrix} 1 & \frac{1}{2} & 0 \end{bmatrix}}_{H_1} x\Big)^3 \\ &+ \underbrace{\begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix}}_{G_2} \Big(\underbrace{\begin{bmatrix} 1 & \frac{1}{2} & 0 \end{bmatrix}}_{H_2} x + 1\Big)^3 \\ &+ \underbrace{\begin{bmatrix} \frac{1}{12} \\ 0 \\ 0 \end{bmatrix}}_{G_3} \Big(\underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{H_3} x\Big)^3 \\ &+ \underbrace{\begin{bmatrix} -\frac{1}{12} \\ 0 \\ 0 \end{bmatrix}}_{G_4} \Big(\underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{H_4} x + 1\Big)^3 + \underbrace{\begin{bmatrix} \frac{1}{12} \\ 0 \\ 0 \end{bmatrix}}_{W} \\ &+ \underbrace{\begin{bmatrix} \frac{1}{3}y_1^3 - \frac{1}{3}(y_1+1)^3 \\ u \\ -y_1^3 \end{bmatrix}}_{g(u,y)}, \; y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x. \end{aligned} \tag{42}$$

A solution of the LMIs (35) is

$$\begin{aligned} P &= \begin{bmatrix} 1.0032 & 0 & 0 \\ 0 & 1.5584 & -0.0309 \\ 0 & -0.0309 & 0.7854 \end{bmatrix}, \\ Y &= \begin{bmatrix} -1.5048 & 0 \\ -2.3417 & -0.4132 \\ 0.8163 & -1.984 \end{bmatrix}, \\ K_1 &= \begin{bmatrix} 2.0068 & 2.7798 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} -4.0131 & -3.2814 \end{bmatrix}, \\ K_3 &= \begin{bmatrix} -12.0397 & -12.5619 \end{bmatrix}, \\ K_4 &= \begin{bmatrix} 12.0397 & 11.5588 \end{bmatrix}, \\ \alpha_1 &= \alpha_2 = \alpha_3 = \alpha_4 = 1.0032. \end{aligned} \tag{43}$$

The following example treats a typical example where a unique square nonlinearity is present in the system dynamics. Therefore, the design proposed in [3], [4] cannot be applied. It will be shown through numerical simulation, that two nonlinear output injection terms are sufficient to build the nonlinear observer.

Example 5: Let us introduce the dynamic system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2^2, \\ \dot{x}_2 &= x_1 + x_2 + x_3 + u, \\ \dot{x}_3 &= x_1 + x_2 - x_3, \\ y_1 &= x_1, \; y_2 = x_2 + x_3. \end{aligned} \tag{44}$$

Since x_2^2 can be rewritten as $\frac{1}{3}(x_2+1)^3 - \frac{1}{3}x_2^3 - x_2 - \frac{1}{3}$ then, the previous system admits the following representation

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} x + \underbrace{\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 0 \end{bmatrix}}_{G_1} \Big(\underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{H_1} x\Big)^3 \\ &+ \underbrace{\begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix}}_{G_2} \Big(\underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{H_2} x + 1\Big)^3 \\ &+ \underbrace{\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 0 \end{bmatrix}}_{W} + \underbrace{\begin{bmatrix} 0 \\ u \\ 0 \end{bmatrix}}_{g(u,y)}, \; y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x. \end{aligned} \tag{45}$$

By solving the LMIs of Theorem 2, we get

$$\begin{aligned} P &= \begin{bmatrix} 0.9570 & 0 & 0 \\ 0 & 0.7962 & -0.0446 \\ 0 & -0.0446 & 1.5741 \end{bmatrix}, \\ Y &= \begin{bmatrix} -2.4164 & 0 \\ 0.2055 & -1.9933 \\ -1.5294 & -0.4010 \end{bmatrix}, \\ K_1 &= \begin{bmatrix} 3.0135 & 2.6094 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} -3.0135 & -3.5665 \end{bmatrix}, \; \alpha_1 = 0.9570, \; \alpha_2 = 0.9570. \end{aligned}$$

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V. CONCLUSION

In this paper the circle-criterion observer design is extended to a broad class of nonlinear systems exhibiting both positive and negative-slope nonlinearities. Motivated by these results, our next contributions shall be focused on the observer-based stabilization problem with multi-objectives designs.