Nonlinear circle-criterion observers in discrete-time

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***Abstract*— Circle-criterion approach to discrete-time nonlinear observer design is presented. The new design method is mainly devoted to either bounded-state nonlinear systems or globally Lipschitz ones. The conditions of existence of globally converging observers are given in terms of a set of numerically tractable linear matrix inequalities. It is shown that the proposed algorithm is able to deal with both nonlinear systems with extremely high Lipschitz constants and nonlinear systems subject to either positive and non-positive slope nonlinearities. The efficacy of the proposed design procedure is testified through numerical examples.**

***Index Terms*— Nonlinear discrete-time observers; Circle criterion; Linear Matrix Inequalities (LMIs).**

I. Introduction

The idea of transforming a nonlinear system into observable canonical forms has been widely used as a key solution to solve nonlinear observation issues, see e.g., [1]. However, the existence of such state transformations that bring the system to some canonical forms of observation are generally attached to complex conditions that cannot always be verified by existing physical systems. In case where the system fails to be put in certain canonical forms, the construction of a high-gain observer turns out to be useful, see e.g., [2], [3], [4], [5], [6], [7]. However, this standard approach which uses a copy of the system dynamics with a unique output correction term may fail due to the limitation of the linear-output-injection term which is basically conceived to defeat the adverse nonlinearities. In our opinion, the conservatism of high-gain observers is mainly due the fact that nonlinearities are viewed as a system uncertainty and their structures are not exploited to reduce the complexity of the observation problem. The reader is referred to [8] for more details on the limitations of feedback in presence of uncertainties and how can the capability of feedback be enhanced if a priori information about the system structure is available. For further details on how to characterize the relation between the distance to unobservability and the Lipschitz constants of nonlinearities, the reader can also see [5], [9], [10] and the references therein.

Besides all these difficulties, discrete-time implementation of high-gain observers is generally raised as a difficult issue since the stability of the observation error cannot be preserved under arbitrary sampling, see e.g., [11], [12], [13]. In [13] the authors established an impossibility theorem that states that the class of uncertain nonlinear systems cannot be stabilized globally by any sampled-data feedback law whenever the sampling rate exceeds the value $\frac{4.757}{\gamma}$ where γ is the slope of the uncertain function. As a result, these particular problems call for wide range of new theories, methodologies and techniques to enable synthesis and stability improvement of sampled-data systems. In this paper, we exploit the circle criterion in discrete-time to give an extension of the works given in references [14], [15] to multi-variables discrete-time nonlinear systems. We shall focus on the design of discrete-time nonlinear observers in an attempt to answer the following question:"Given a discrete-time nonlinear system with either positive and non-positive slope nonlinearities, how to exploit the structure of nonlinearities in order to set up a converging observer with less conservative conditions". To answer this question, the developed design method in discrete-time slightly differs from that developed in the continuous-time case [15], in the sense that, either positive and non-positive slope nonlinearities are tolerated. First, we begin by analyzing the discrete-time circle criterion observer for globally Lipschitz systems. Subsequently, we focus on the design of nonlinear observers for bounded-state systems whose nonlinearities have not a priori bounded slopes. Motivated by the results given in [16], we derive LMI-based conditions that ensures the existence of a semi-globally convergent observer for the bounded state system. The main difference between our discrete-time design method and that proposed in continuous-time [16], is that the system being considered is not in certain canonical forms and nonlinearities are saturated by a *new smooth saturation function* that preserves the differentiability of the saturated functions. Finally, illustrative examples are given to highlight the efficacy and the main features of the circle-criterion observers in discrete-time.

Notations. Throughout this paper, we note by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the set of natural numbers, the set of integer numbers and the set of real numbers, respectively. The notation $A > 0$ (resp. $A < 0$) means that the matrix A is positive definite (resp. negative definite). A' is the matrix transpose of A. "$\star$" is used to notify an element which is induced by transposition. $\triangleq$ stands for an equality by definition. $\circ$ stands for the composition operator of functions. $f^{(-1)}(x)$ is the inverse function of the scalar function $f(x)$. $|\cdot|$ stands for the absolute value.

II. Circle criterion observer design in discrete-time

Consider the nonlinear discrete-time system

$$\begin{aligned} x_{k+1} &= A\,x_k + \sum_{i=1}^{\mu} G_i f_i(H_i\,x_k) + \psi(u_k, y_k), \\ y_k &= C\,x_k, \end{aligned} \tag{1}$$

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where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the system input and $y_k \in \mathbb{R}^p$ is the system measured output. The nominal matrices $A \in \mathbb{R}^{n\times n}$, $(G_i)_{1\leq i\leq \mu} \in \mathbb{R}^{n\times 1}$, $(H_i)_{1\leq i\leq \mu} \in \mathbb{R}^{1\times n}$ and $C \in \mathbb{R}^{p\times n}$ are constant known matrices where (A, C) is an observable pair. The term $\psi(u_k, y_k)$ is an arbitrary real-valued vector that depends on the system inputs and outputs. $(f_i(H_i\, x_k))_{1\leq i\leq \mu}$ are $\mathscr{C}^1$ state-dependent nonlinearities verifying the following growth conditions

$$\frac{\mathrm{d}f_i(s)}{\mathrm{d}s} \geq 0,\ 1 \leq i \leq \mu,\ \ \forall s \in \mathbb{R}. \tag{2}$$

Systems of form (1) may represent the dynamics of pure[1] discrete-time systems or sampled systems issued from continuous-time systems studied in reference [15]. However, the new representation (1) may include more general nonlinearities which may not have positive slopes, see example 2 for more details. Since the limitation imposed by sampling cannot be removed in discrete-time, in this paper, we show how to exploit the system nonlinearities in order to set up a converging observer for system (1) under the assumption that the slope of nonlinearities do not exhibit an escape to infinity in finite time. Therefore, two different classes of systems are studied: globally Lipschitz systems and bounded-state nonlinear systems that have not a priori bounded slopes. If the slopes of nonlinearities escape to infinity in finite time, the developed observation procedure shall be valid in large bounded set that can be a priori estimated.

In this section, we show how to conceive converging observers by employing multiple-output-injection terms. This idea permits to exploit each nonlinearity in observer design without making any severe assumption on the whole vector nonlinearity. The result is summarized in the following statement.

Theorem 1: Consider the nonlinear system (1) and assume that $\left(\left|\frac{\mathrm{d}f_i(s)}{\mathrm{d}s}\right|\right)_{1\leq i\leq\mu} < \infty$ for all $s \in \mathbb{R}$. Let $(\beta_i)_{1\leq i\leq \mu}$ and $(\varrho_{\min}(i))_{1\leq i\leq \mu}$ be two sets of positive constants such that

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\Big(f_i(s) + \beta_i\, s\Big)\right)^{-1} > \varrho_{\min}(i),\ \forall s \in \mathbb{R},\ \ 1 \leq i \leq \mu. \tag{3}$$

If there exist a symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$, a constant matrix $Y \in \mathbb{R}^{n\times p}$ and a set of row vectors $(K_i)_{1\leq i\leq \mu} \in \mathbb{R}^p$ such that the following linear matrix inequalities hold

$$\begin{aligned}
&(\mathscr{C}_1)\ \begin{bmatrix} -P & A'P - \sum_{i=1}^{\mu}\beta_i\, H_i' G_i' P + C'Y' \\ \star & -P \end{bmatrix} < 0,\\
&(\mathscr{C}_2)\ G_i' P\Big(A - \sum_{i=1}^{\mu}\beta_i\, G_i H_i\Big) + G_i' Y C = \\
&\qquad -\frac{\mu}{2}(H_i + K_i C),\ \ 1 \leq i \leq \mu,\\
&(\mathscr{C}_3)\ G_i' P G_i - \varrho_{\min}(i) \leq 0,\ \ 1 \leq i \leq \mu,
\end{aligned} \tag{4}$$

[1]It means systems that are discrete in nature.

then, $\lim_{k\to\infty} x_k - \hat{x}_k = 0$, where $\hat{x}_k$ is the state vector of the nonlinear discrete-time observer

$$\begin{aligned}
\hat{x}_{k+1} &= A\,\hat{x}_k + \sum_{i=1}^{\mu} G_i\, f_i\Big(H_i\,\hat{x}_k + K_i(C\hat{x}_k - y_k)\Big)\\
&\quad + \psi(u_k, y_k) + \sum_{i=1}^{\mu}\beta_i G_i K_i (C\hat{x}_k - y_k)\\
&\quad + P^{-1}Y\big(C\hat{x}_k - y_k\big).
\end{aligned} \tag{5}$$

Proof. Let $\mathscr{G}_i(s_k) \triangleq f_i(s_k) + \beta_i\, s_k,\ 1 \leq i \leq \mu$. Then, system (32) and observer (35) can be rewritten respectively as follows

$$\begin{aligned}
x_{k+1} &= \Big(A - \sum_{i=1}^{\mu}\beta_i\, G_i H_i\Big)\, x_k + \sum_{i=1}^{\mu} G_i \mathscr{G}_i(H_i\, x_k)\\
&\quad + \psi(u_k, y_k),\\
y_k &= C\, x_k,\ \ (x_k, u_k) \in \Omega \times \mathscr{U},
\end{aligned} \tag{6}$$

$$\begin{aligned}
\hat{x}_{k+1} &= \Big(A - \sum_{i=1}^{\mu}\beta_i\, G_i H_i\Big)\,\hat{x}_k\\
&\quad + \sum_{i=1}^{\mu} G_i \mathscr{G}_i(H_i\,\hat{x}_k + K_i(C\hat{x} - y_k))\\
&\quad + \psi(u_k, y_k) + P^{-1}Y(C\hat{x}_k - y_k).
\end{aligned} \tag{7}$$

Let $A_c \triangleq A - \sum_{i=1}^{\mu}\beta_i\, G_i H_i$. Then, if we note the observation error as $e_k = x_k - \hat{x}_k$. This implies that

$$\begin{aligned}
e_{k+1} &= \Big(A_c + P^{-1}YC\Big)\, e_k + \sum_{i=1}^{\mu} G_i \mathscr{G}_i(H_i\, x_k)\\
&\quad - \sum_{i=1}^{\mu} G_i \mathscr{G}_i(H_i\,\hat{x}_k + K_i(C\hat{x}_k - y_k))
\end{aligned} \tag{8}$$

Using the mean-value Theorem, we have for a given scalar $\mathscr{C}^{(1)}$-function $\varphi(\cdot)$

$$\varphi(v) - \varphi(w) = \int_0^1 \frac{\partial\varphi(s)}{\partial s}\bigg|_{s=v-\lambda(v-w)} (v - w)\,\mathrm{d}\,\lambda. \tag{9}$$

Then, if we put

$$\begin{aligned}
v_i(k) &\triangleq H_i\, x_k,\\
w_i(k) &\triangleq H_i\,\hat{x}_k + K_i(C\hat{x}_k - y_k),\\
\omega_i(k) &\triangleq v_i(k) - \lambda(v_i(k) - w_i(k)).
\end{aligned} \tag{10}$$

the observation error dynamics can be rewritten as

$$\begin{aligned}
e_{k+1} &= \Big(A_c + P^{-1}YC\Big)\, e_k\\
&\quad + \int_0^1 \sum_{i=1}^{\mu} G_i \frac{\partial\mathscr{G}_i(s_k)}{\partial s_k}\Big|_{s_k=\omega_i(k)}\Big(H_i + K_i C\Big)\, e_k\,\mathrm{d}\,\lambda\\
&= \int_0^1 \Big(A_c + P^{-1}YC\Big)\, e_k\,\mathrm{d}\,\lambda\\
&\quad + \int_0^1 \sum_{i=1}^{\mu} G_i \frac{\partial\mathscr{G}_i(s_k)}{\partial s_k}\Big|_{s_k=\omega_i(k)}\Big(H_i + K_i C\Big)\, e_k\,\mathrm{d}\,\lambda.
\end{aligned} \tag{11}$$

By taking the Lyapunov function $V_k = e'_k P e_k$. Then, we obtain

$$\begin{aligned} & V_{k+1} - V_k = e'_{k+1} P e_{k+1} - e'_k P e_k \\ & = \Bigg[\int_0^1 \Big(A_c + P^{-1} Y C \Big) e_k \, \mathrm{d}\,\lambda \\ & + \int_0^1 \sum_{i=1}^{\mu} G_i \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} \Big(H_i + K_i C \Big) e_k \mathrm{d}\,\lambda \Bigg]' \\ & P \times \Bigg[\int_0^1 \Big(A_c + P^{-1} Y C \Big) e_k \, \mathrm{d}\,\lambda \\ & + \int_0^1 \sum_{i=1}^{\mu} G_i \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} \Big(H_i + K_i C \Big) e_k \, \mathrm{d}\lambda \Bigg] \\ & - e'_k P e_k. \end{aligned} \tag{12}$$

Since for any constant symmetric matrix $M \in \mathbb{R}^{n \times n}$, $M = M' > 0$, scalar $\gamma > 0$, vector function $\omega : [0,\ \gamma] \mapsto \mathbb{R}^n$ such that the integration in the following is well defined, we have $\gamma \int_0^{\gamma} \omega'(\alpha) M \omega(\alpha) \mathrm{d}\alpha \geq \left(\int_0^{\gamma} \omega(\alpha) \mathrm{d}\alpha \right)' M \left(\int_0^{\gamma} \omega(\alpha) \mathrm{d}\alpha \right)$. Then, we can then write

$$\begin{aligned} & V_{k+1} - V_k \leq \int_0^1 \Bigg[\Big(A_c + P^{-1} Y C \Big) e_k \\ & + \sum_{i=1}^{\mu} G_i \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} \Big(H_i + K_i C \Big) e_k \Bigg]' \\ & P \times \Bigg[\Big(A_c + P^{-1} Y C \Big) e_k \\ & + \sum_{i=1}^{\mu} G_i \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} \Big(H_i + K_i C \Big) e_k \Bigg] \mathrm{d}\,\lambda \\ & - \int_0^1 e'_k P e_k \, \mathrm{d}\,\lambda. \end{aligned} \tag{13}$$

By expanding the right-hand side of the last inequality, the difference $\Delta V_k \triangleq V_{k+1} - V_k$ is bounded as follows

$$\begin{aligned} \Delta V_k \leq & \int_0^1 e'_k (A_c + P^{-1} Y C)' P (A_c + P^{-1} Y C) \, e_k \, \mathrm{d}\,\lambda \\ & - \int_0^1 e'_k P e_k \, \mathrm{d}\,\lambda \\ & + 2 \int_0^1 \sum_{i=1}^{\mu} \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} e'_k \Big(H_i + K_i C \Big)' \\ & G'_i P \Big(A_c + P^{-1} Y C \Big) e_k \mathrm{d}\,\lambda \\ & + \int_0^1 \Bigg[\sum_{i=1}^{\mu} \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} e'_k \Big(H_i + K_i C \Big)' G'_i \Bigg] P \times \\ & \Bigg[\sum_{i=1}^{\mu} \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} G_i \Big(H_i + K_i C \Big) e_k \Bigg] \mathrm{d}\,\lambda. \end{aligned} \tag{14}$$

By the Cauchy-Schwartz inequality,

$$\begin{aligned} & \mu \sum_{i=1}^{\mu} a'_i P a_i \geq \sum_{i=1}^{\mu} a'_i P \Big(\sum_{i=1}^{\mu} a_i \Big), \\ & a_i \in \mathbb{R}^n,\ P \in \mathbb{R}^{n \times n}. \end{aligned} \tag{15}$$

Then, we can write that

$$\begin{aligned} & \int_0^1 \Bigg[\sum_{i=1}^{\mu} \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} e'_k \Big(H_i + K_i C \Big)' G'_i \Bigg] P \times \\ & \Bigg[\sum_{i=1}^{\mu} \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} G_i \Big(H_i + K_i C \Big) e_k \Bigg] \mathrm{d}\,\lambda \\ & \leq \mu \int_0^1 \sum_{i=1}^{\mu} \left(\frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} \right)^2 e'_k \Big(H_i + K_i C \Big)' \times \\ & G'_i P G_i \Big(H_i + K_i C \Big) e_k \, \mathrm{d}\,\lambda. \end{aligned} \tag{16}$$

This implies that if the following holds

$$\begin{aligned} \Delta V_k \leq & \int_0^1 e'_k (A_c + P^{-1} Y C)' P (A_c + P^{-1} Y C) \, e_k \, \mathrm{d}\,\lambda \\ & - \int_0^1 e'_k P e_k \, \mathrm{d}\,\lambda \\ & + 2 \int_0^1 \sum_{i=1}^{\mu} \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} e'_k \Big(H_i + K_i C \Big)' \\ & G'_i P \Big(A_c + P^{-1} Y C \Big) e_k \mathrm{d}\,\lambda \\ & + \mu \int_0^1 \sum_{i=1}^{\mu} \left(\frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} \right)^2 e'_k \Big(H_i + K_i C \Big)' \\ & \times G'_i P G_i \Big(H_i + K_i C \Big) e_k \, \mathrm{d}\,\lambda \end{aligned} \tag{17}$$

then (14) holds. Let us choose P such that

$$(A_c + P^{-1} Y C)' P (A_c + P^{-1} Y C) - P = -Q < 0, \quad Q > 0. \tag{18}$$

or equivalently (by the Schur complement),

$$\begin{bmatrix} -P & A'P - \sum_{i=1}^{\mu} \beta_i \, H'_i G'_i P + C'Y' \\ \star & -P \end{bmatrix} < 0. \tag{19}$$

By adding the following equality constraint

$$G'_i P A_c + G'_i Y C = -\frac{\mu}{2} (H_i + K_i C), \quad 1 \leq i \leq \mu, \tag{20}$$

then, we obtain

$$\begin{aligned} \Delta V_k \leq & -e'_k \, Q \, e_k - \mu \int_0^1 \sum_{i=1}^{\mu} \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} \\ & \times e'_k \Big(H_i + K_i C \Big)' \Big(H_i + K_i C \Big) e_k \, \mathrm{d}\,\lambda \\ & + \mu \int_0^1 \sum_{i=1}^{\mu} \left(\frac{\partial \mathscr{G}_i(s_k)}{\partial s_k} \Big|_{s_k = \omega_i(k)} \right)^2 \\ & e'_k \Big(H_i + K_i C \Big)' G'_i P G_i \Big(H_i + K_i C \Big) e_k \, \mathrm{d}\,\lambda. \end{aligned} \tag{21}$$

Since $\varrho_{\min}(i) < \left(\frac{\partial \mathscr{G}_i(s_k)}{\partial s_k}\Big|_{s_k=\omega_i(k)} \right)^{-1}$, $1 \le i \le \mu$, and $G_i' P G_i - \varrho_{\min}(i) \le 0$, $1 \le i \le \mu$. Then,

$$\begin{aligned}\Delta V_k &\le -e_k' \, Q \, e_k \\ &\quad - \mu \int_0^1 \sum_{i=1}^{\mu} \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k}\Big|_{s_k=\omega_i(k)} e_k' \Big(H_i + K_i C \Big)' \\ &\quad \times \Big[1 - G_i' P G_i \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k}\Big|_{s_k=\omega_i(k)} \Big] \Big(H_i + K_i C \Big) \, e_k \, \mathrm{d}\,\lambda \\ &\quad \le -e_k' \, Q \, e_k \le 0. \end{aligned} \tag{22}$$

Since $Q > 0$ then, the observation error is globally asumptotically stable. This ends the proof.

Remark 1: For globally Lipschitz systems where $\left|\frac{\mathrm{d} f_i(s)}{\mathrm{d}s}\right| < \infty$, $1 \le i \le \mu$, $\forall s \in \mathbb{R}$, condition (2) is not necessary for the the observer design. Condition (3) suffices for the determination of the coefficients $(\beta_i)_{1 \le i \le \mu}$.

Remark 2: The coefficient $(\beta_i)_{1 \le i \le \mu}$ are basically introduced in order to make the smooth functions $\frac{\mathrm{d} f_i(s)}{\mathrm{d}s} + \beta_i$, $1 \le i \le \mu$ strictly positive. In the meantime, the coefficients $(\varrho_{\min}(i))_{1 \le i \le \mu}$ shall be maximized in order to make the LMIs of Theorem 1 feasible. Therefore, $(\beta_i)_{1 \le i \le \mu}$ must be chosen as small as possible.

It is worth to mention that from Eq. (11), the observation error dynamics can be rewritten as

$$\begin{aligned} e_{k+1} &= \Big(A_c + P^{-1} Y C \Big) \, e_k + \sum_{i=1}^{\mu} G_i \varphi_i(k, z_i(k)), \\ z_i(k) &= \big(H_i + K_i C \big) \, e_k, \; 1 \le i \le \mu, \end{aligned} \tag{23}$$

where $A_c + P^{-1} Y C$ is a stable matrix, and $\varphi_i(k, z_i(k)) \triangleq \int_0^1 \frac{\partial \mathscr{G}_i(s_k)}{\partial s_k}\Big|_{s_k=\omega_i(k)} z_i(k) \, \mathrm{d}\,\lambda$. According to (23), the observer design problem is equivalent to a stabilization of a linear discrete-time system interconnected with a sum of memoryless nonlinearities verifying the sector conditions $z_i \varphi_i(k, z_i(k)) \ge 0$; $\forall i$.

Example 1: Consider the continuous-time nonlinear system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + \frac{\gamma}{2} \sin\big(x_1(t) + x_2(t) \big) + u(t), \\ \dot{x}_2(t) &= \gamma \sin\big(x_1(t) + x_2(t) \big) + u(t), \\ y(t) &= x_1(t), \end{aligned} \tag{24}$$

where γ is a positive real constant. The Lipschitz constant of the aforementioned system is equal to 2γ. By taking the Euler discrete-time model with sampling period δ, we obtain

$$\begin{aligned} x_{k+1} &= A \, x_k + \delta \, \gamma \, G_1 \sin(H_1 \, x_k) + \delta \, B \, u_k, \\ y_k &= C \, x_k \end{aligned} \tag{25}$$

where

$$\begin{aligned} F &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; A = (I + \delta \, F), G_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; H_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}'. \end{aligned} \tag{26}$$

In order to apply the result of Theorem 1, let us rewrite the dynamics of system (25) as follows

$$\begin{aligned} x_{k+1} &= A_c \, x_k + G_1 \Big[\delta \, \gamma \sin\big(H_1 \, x_k \big) + \beta_1 H \, x_k \Big] + \delta \, B \, u_k, \\ y_k &= C \, x_k \end{aligned} \tag{27}$$

where $A_c = A - \beta_1 G_1 \, H_1 = \begin{bmatrix} 1 - \frac{1}{2}\beta_1 & \delta - \frac{1}{2}\beta_1 \\ -\beta & 1 - \beta_1 \end{bmatrix}$, and $\beta_1 \triangleq \frac{3}{2}\delta\gamma$, $\mathscr{G}_1(s) \triangleq \delta\gamma\Big[\sin(s) + \frac{3}{2} s \Big]$. Here, $\frac{\mathrm{d}\mathscr{G}_1(s)}{\mathrm{d}s} = \delta \, \gamma[\cos(s) + \frac{3}{2}] > 0$. Consequently, we can choose $\varrho_{\min} = \frac{2}{5\delta\gamma}$. The objective of introducing this example is to show that the LMIs of Theorem 1 are not conservative when the value of γ increases. Therefore, we shall check the solvability of LMIs (4) for increasing values of γ. The results are given in the following table for $\delta = 0.01$.

γ	P	Y	K
1	$\begin{bmatrix} 62.421 & -13.249 \\ -13.249 & 6.0711 \end{bmatrix}$	$\begin{bmatrix} -36.07 \\ 0.63377 \end{bmatrix}$	-1.8687
5	$\begin{bmatrix} 21.891 & -4.5643 \\ -4.5643 & 1.9313 \end{bmatrix}$	$\begin{bmatrix} -10.841 \\ 0.083439 \end{bmatrix}$	-2.663
10	$\begin{bmatrix} 22.168 & -5.3463 \\ -5.3463 & 2.5237 \end{bmatrix}$	$\begin{bmatrix} -9.1738 \\ 0.0054277 \end{bmatrix}$	-2.4967
50	$\begin{bmatrix} 48.486 & -22.881 \\ -22.881 & 11.429 \end{bmatrix}$	$\begin{bmatrix} -0.56831 \\ 0.3474 \end{bmatrix}$	-2.8462
5000	$\begin{bmatrix} 3898.5 & -1949.3 \\ -1949.3 & 974.7 \end{bmatrix}$	$\begin{bmatrix} -92.064 \\ 47.633 \end{bmatrix}$	-3.0065
40000	$\begin{bmatrix} 33885 & -16943 \\ -16943 & 8471.3 \end{bmatrix}$	$\begin{bmatrix} -648.94 \\ 325.95 \end{bmatrix}$	-3.0015

For the above values, the states of the following observer

$$\begin{aligned} \hat{x}_{k+1} &= A_c \, \hat{x}_k + G_1 \Big[\delta \, \gamma \sin\big(H_1 \hat{x}_k + K_1 (C \hat{x}_k - y_k) \big) \\ &\quad + \beta \Big(H_1 \, \hat{x}_k + K_1 (C \hat{x}_k - y_k) \Big) \Big] \\ &\quad \delta \, B \, u_k + P^{-1} Y \big(C \hat{x}_k - y_k \big), \end{aligned} \tag{28}$$

converge asymptotically to the states of system (25) for any initial conditions $\hat{x}_0 \in \mathbb{R}^n$. One of the main features of the developed technique is that the observer gains are not high-gain vectors as γ increases which implies that the new proposed design improves the transient behaviors of the estimates.

III. Bounded-state nonlinear systems

In this section we extend the design of the circle-criterion observers to bounded-state systems whose nonlinearities can be seen as globally Lipschitz in a large compact set that belongs to $\mathbb{R}^n$. Before giving the main result of this paper, we begin by exposing the following important result.

Lemma 1: Consider the saturation function $\mathscr{S}(v)$ defined as

$$\mathscr{S}(v) \triangleq \begin{cases} v & \text{if}, -\rho \le v \le \rho, \\ \rho + (v - \rho) \, e^{\rho - v} & \text{if}, v > \rho, \\ -\rho + (v + \rho) \, e^{\rho + v} & \text{if}, v < -\rho. \end{cases} \tag{29}$$

Then, $\mathscr{S}(v)$ and $\frac{\mathrm{d}}{\mathrm{d}v}\mathscr{S}(v)$ are bounded and continuous over $\mathbb{R}$.

Proof. The proof is omitted for space limitation.

Consider now system (1) where all the system states are assumed to be bounded for a given initial condition $x_0 \in \Omega \subset \mathbb{R}^n$ and $u_k \in \mathscr{U} \subset \mathbb{R}^m$. Using the result of Lemma 1, we can always find a set of positive constants $(\rho_i)_{1\le i\le\mu}$ and a set of real numbers $(\tau_i)_{1\le i\le\mu}$ such that

$$\begin{aligned} f_i(H_i\,x_k) &= \mathscr{S}_i \circ f_i(H_i\,x_k),\ 1\le i\le\mu,\ x_k\in\Omega,\\ f_i(\tau_i) &= \rho_i,\ 1\le i\le\mu, \end{aligned} \tag{30}$$

where

$$\mathscr{S}_i(v) \triangleq \begin{cases} v & \text{if}, -\rho_i \le v \le \rho_i,\\ \rho_i + (v-\rho_i)\,e^{\rho_i - v} & \text{if}, v > \rho_i,\\ -\rho_i + (v+\rho_i)\,e^{\rho_i + v} & \text{if}, v < -\rho_i. \end{cases} \tag{31}$$

Consequently, system (1) can be rewritten in the following form

$$\begin{aligned} x_{k+1} &= A\,x_k + \sum_{i=1}^{\mu} G_i \mathscr{S}_i \circ f_i(H_i\,x_k) + \psi(u_k, y_k),\\ y_k &= C\,x_k,\ \ (x_k, u_k) \in \Omega \times \mathscr{U}. \end{aligned} \tag{32}$$

Thanks to the developed saturation functions $(\mathscr{S}_i(v))_{1\le i\le\mu}$, the bounded state system (1) is viewed as a *smooth dynamical system* with bounded nonlinearities. The employed saturation functions $(\mathscr{S}_i(v))_{1\le i\le\mu}$ approach the classical non-differentiable saturation functions

$$\mathrm{Sat}_i(v) \triangleq \begin{cases} v & \text{if}, -\rho_i \le v \le \rho_i,\\ \rho_i & \text{if}, v > \rho_i,\\ -\rho_i & \text{if}, v < -\rho_i. \end{cases} \tag{33}$$

when $|v| \gg (\rho_i)_{1\le i\le\mu}$. In this section, the new equivalent structure of system (32) is exploited to build converging observers that enjoy the properties to be smooth too. The design of the observer is given by the following statement.

Corollary 1: Consider system (1) under assumption (2). Define $\Omega \triangleq \{x_k \in \mathbb{R}^n \mid |x_i(k)| \le \alpha_i,\ 1\le i\le n\}$ with $\alpha_i > 0,\ 1 \le i \le n$. Assume that for all bounded input $u_k \in \mathscr{U} \subset \mathbb{R}^m$, and some initial conditions $x_0 \in \Omega$ the state vector x_k belongs to the same subset Ω for all $k \in \mathbb{Z}_{>0}$. Let $(\rho_i)_{1\le i\le\mu}$ be positive saturation levels defined as in (30)-(31). If we choose two sets of positive constants $(\beta_i)_{1\le i\le\mu}$ and $(\varrho_{\min}(i))_{1\le i\le\mu}$ such that for all $1\le i\le\mu$, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathscr{S}_i \circ f_i(s) + \beta_i\,s\Big)\right)^{-1} > \varrho_{\min}(i),\ \forall s \in \mathbb{R}, \tag{34}$$

and there exist a symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$, a constant matrix $Y \in \mathbb{R}^{n\times p}$ and a set of row vectors $(K_i)_{1\le i\le\mu} \in \mathbb{R}^p$ such that the conditions $(\mathscr{C}_1)$, $(\mathscr{C}_2)$ and $(\mathscr{C}_3)$ of Theorem 1 hold then, for any initial condition $\hat{x}_0$, the states of the following observer

$$\begin{aligned} \hat{x}_{k+1} &= A\,\hat{x}_k + \sum_{i=1}^{\mu} G_i \mathscr{S}_i \circ f_i\Big(H_i\,\hat{x}_k + K_i(C\hat{x}_k - y_k)\Big)\\ &\quad + \psi(u_k, y_k)\\ &\quad + \sum_{i=1}^{\mu} \beta_i G_i K_i (C\hat{x}_k - y_k) + P^{-1} Y\big(C\hat{x}_k - y_k\big). \end{aligned} \tag{35}$$

converge asymptotically to the states of system (1).

Proof. The proof is omitted here because it is quite similar to the proof of Theorem 1. The only difference in the proof is that $\mathscr{G}_i(s_k)$ becomes equal to $\mathscr{S}_i(s_k) \circ f_i(s_k) + \beta_i\,s_k$. This ends the proof.

Remark 3: A practical method to determine the coefficient $(\beta_i)_{1\le i\le\mu}$ for given saturation level $(\rho_i)_{1\le i\le\mu}$, is to see by how much the functions

$$\begin{aligned} g_i(s) &\triangleq \frac{\mathrm{d}f_i(s)}{\mathrm{d}s}\,e^{\rho_i} - |f_i(s)|\Big(1 - |f_i(s)| + \rho_i\Big),\\ &\quad |f_i(s)| > \rho_i,\ 1 \le i \le \mu, \end{aligned} \tag{36}$$

drop below zero. The coefficients $(\beta_i)_{1\le i\le\mu}$ are determined as the minimum values that make $g_i(s) + \beta_i > 0$ for all i.

From the LMI conditions of Corollary 1, we realize that, if the matrix P verifies the conditions $G_i' P G_i = 0$ for all i then, the breakdown of the observer will be independent from the slopes of nonlinearities. However, if conditions $G_i' P G_i = 0$, $1 \le i \le \mu$ are imposed, the positive definite requirement of P should be weakened to positive semi-definite. As a result, the linear output injection term cannot be computed through $P^{-1}Y$ since P may not be invertible. For this particular reason, these conditions are not considered herein. However, the conditions $G_i' P G_i \le \varepsilon_i$, $1\le i\le\mu$ can be imposed for small values of ε_i which means that the slopes of nonlinearities are maximized. Hence, the domain of observation can be set as large as possible.

Corollary 2: Consider system (1). If there exist a symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$, a constant matrix $Y \in \mathbb{R}^{n\times p}$, a set of row vectors $(K_i)_{1\le i\le\mu} \in \mathbb{R}^p$ and a set of positive constants $(\varepsilon_i)_{1\le i\le\mu}$ such that the following generalized eigenvalue problem is feasible

$$\min_{P, Y, K_i} \varepsilon_i,\ \ 1\le i\le\mu,$$

subject to

$$\begin{aligned} &\begin{bmatrix} -P & A'P - \sum_{i=1}^{\mu} \beta_i\,H_i' G_i' P + C'Y' \\ \star & -P \end{bmatrix} < 0,\\ &G_i' P\Big(A - \sum_{i=1}^{\mu} \beta_i\,G_i H_i\Big) + G_i' Y C\\ &\quad = -\frac{\mu}{2}(H_i + K_i C),\ \ 1\le i\le\mu,\\ &G_i' P G_i - \varepsilon_i \le 0,\ \ 1\le i\le\mu, \end{aligned} \tag{37}$$

then, there exist a set of saturation levels $(\rho_i)_{1\le i\le\mu}$ and $(\beta_i)_{1\le i\le\mu}$ such that the states of observer (35) converge asymptotically to those of system (1) whenever the states of (1) do not leave the set $\mathscr{D}$ defined as $\mathscr{D} \triangleq \Bigg\{ x_k \in \mathbb{R}^n \mid \big|H_i\,x_k\big| \le \left(\frac{\mathrm{d}\mathscr{G}_i(s)}{\mathrm{d}s}\right)^{(-1)}\Bigg|_{s=\frac{1}{\varepsilon_i}}, 1 \le i \le \mu,\quad k \in \mathbb{Z}_{\ge 0} \Bigg\}$.

Proof. The result of this corollary is a direct consequence of Corollary 1. The result of Corollary 2 shows the inverse design of Corollary 1 when the slopes of nonlinearities are put as LMIs variables.

Example 2: In order to show that the presented algorithm can deal with positive and non positive slopes nonlinearities, let us consider the following nonlinear system

$$\dot{x}(t) = \begin{bmatrix} 0 & 10 & 0 \\ -10 & 0 & 5 \\ 0 & -\frac{10}{3} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_3^2(t) \\ x_3^3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t),\ y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(t). \tag{38}$$

By taking $f(x(t)) = \begin{bmatrix} x_3^2(t) \\ x_3^3(t) \end{bmatrix}$, system (38) does not verify the condition $\left(\frac{\partial f(x(t))}{\partial x(t)}\right)' + \left(\frac{\partial f(x(t))}{\partial x(t)}\right) \geq 0$. Therefore, the design proposed in [15] cannot be applied. The Euler discrete-time approximation of the last system gives

$$\begin{aligned} x_{k+1} &= \left(I + \delta \begin{bmatrix} 0 & 10 & 0 \\ -10 & 0 & 5 \\ 0 & -\frac{10}{3} & 0 \end{bmatrix} \right) x_k \\ &\quad + \delta \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_3^2(k) \\ x_3^3(k) \end{bmatrix} + \delta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_k, \\ y_k &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x_k. \end{aligned} \tag{39}$$

Here, the system nonlinearity $x_3^2(k)$ has not always a positive slope. However, by expanding the square nonlinearity $x_3^2(k)$ as follows

$$\begin{aligned} x_3^2(k) &= (x_3^2(k) + 2\,x_3^3(k) + x_3(k)) - 2x_3^3(k) - x_3(k), \\ &= f_1(x_3(k)) - 2f_2(x_3(k)) - f_3(x_3(k)), \end{aligned} \tag{40}$$

where $f_1(s) = 2\,s^3 + s^2 + s$, $f_2(s) = s^3$ and $f_3(s) = s$. Then, $f_1(s)$, $f_2(s)$ and $f_3(s)$ have all positive slopes for all $s \in \mathbb{R}$. According to this decomposition, system (39) is rewritten as follows

$$\begin{aligned} x_{k+1} &= \underbrace{\left(I + \delta \begin{bmatrix} 0 & 10 & -1 \\ -10 & 0 & 5 \\ 0 & -\frac{10}{3} & 0 \end{bmatrix} \right)}_{A} x_k \\ &\quad + \delta \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{G_1} \left(2\,x_3^3(k) + x_3^2(k) + x_3(k) \right) \\ &\quad + \delta \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{G_2} x_2^3(k),\ y_k = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{C} x_k. \end{aligned} \tag{41}$$

IV. Conclusion

Circle-criterion observers for nonlinear discrete-time systems with positive-slope and negative-slope nonlinearities are developed. We showed that the existence of such observers is conditioned by the solutions of a set of linear matrix inequality conditions with equality constraints. The present work can be seen as an extension of existing works on discrete-time Luenberger observers and a counterpart of the work on the circle criterion observer developed in the continuous-time case.

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